Modeling Solar Energetic Particles from the Sun to Earth

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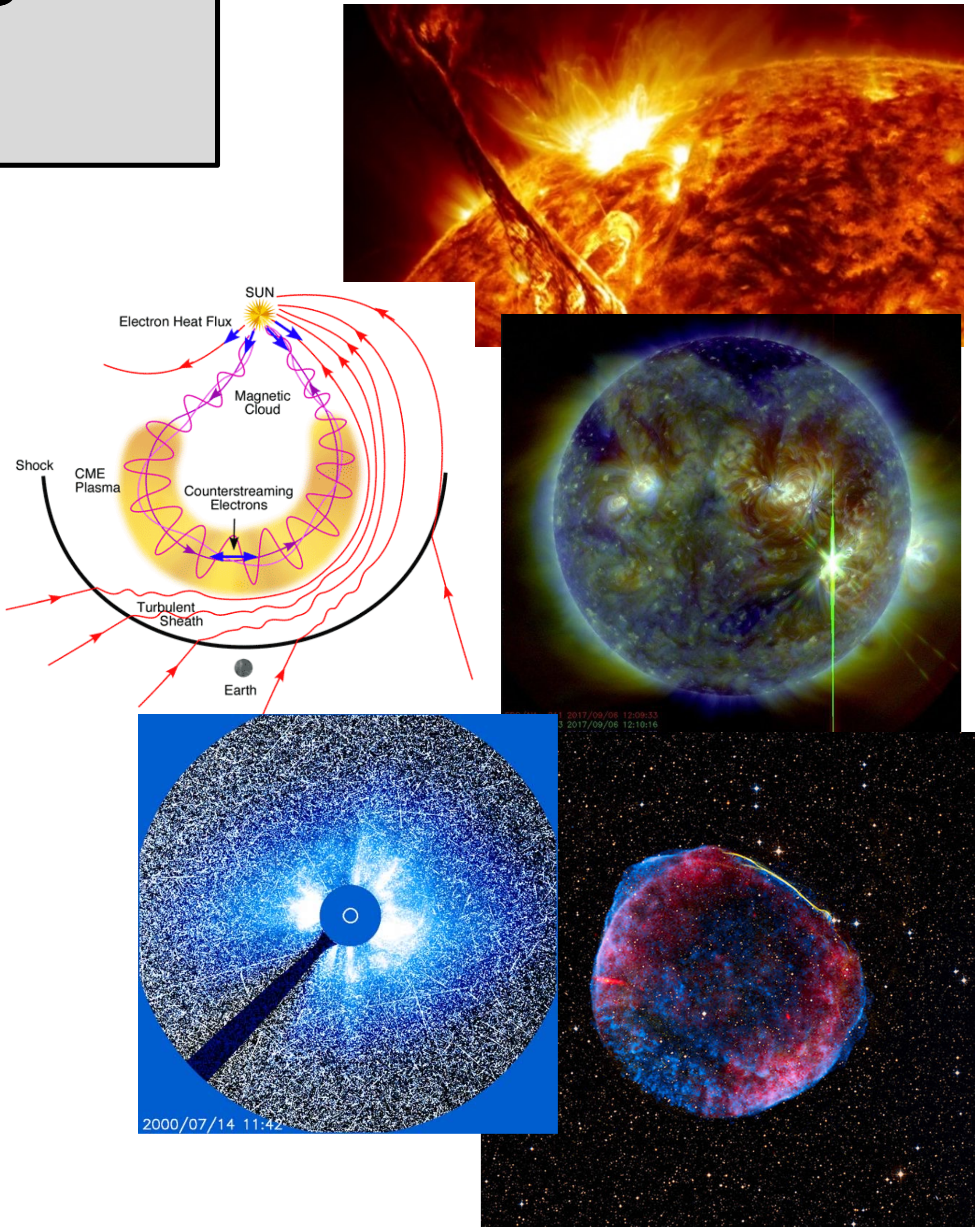
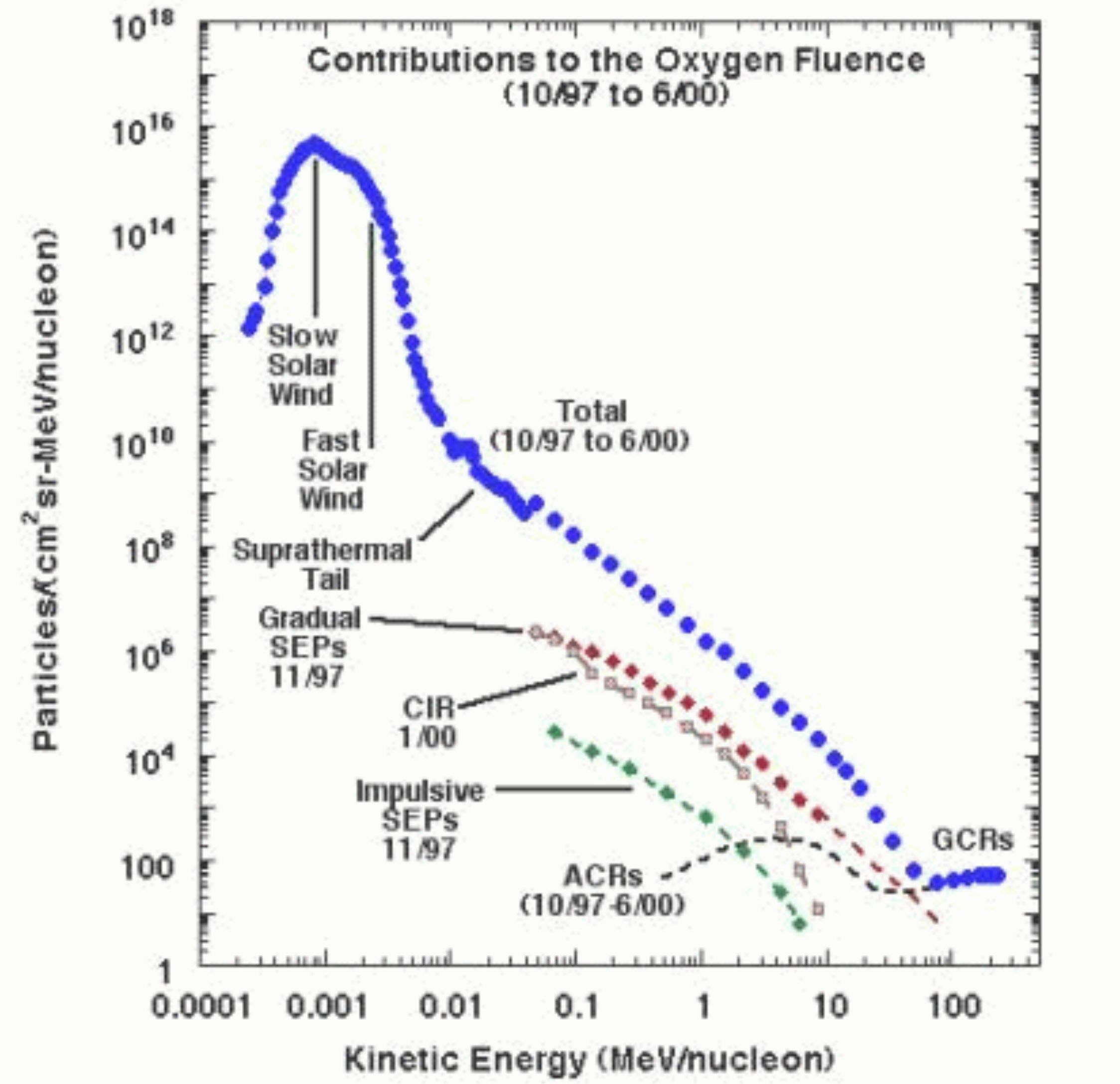
Why Study Energetic Particles from the Sun?

- The Sun is significant source of energetic particles and provides an excellent target for studying the underlying physics of particle acceleration – a fundamental topic In astrophysics
- This physics has significant overlap and application to astrophysical plasmas, such as supernovae remnants, which produce the majority of galactic cosmic rays, but for which we cannot study *in situ*.
- Characteristic energies at CME-driven shocks can exceed a few MeV, with maximum energies up to a few GeV.   Sufficient to study the acceleration process.



High-energy charged particles in the solar system

- Galactic cosmic rays(GCRs)
- Anomalous cosmic rays (ACRs)
- Solar energetic particles (SEPs)
  - High-intensity, long duration events associated with coronal mass ejections (Gradual SEPs)
  - Lower intensity, short lived events associated with brief, impulsive solar flares (Impulsive SEPs)
  - Recurrent events associated with solar wind structures that co-rotate with the Sun (CIRs)
  - Solar Proton Events (SPEs) are a subclass of SEPs associated the highest intensity events that are defined by the NOAA/GOES.
  - Ground-level enhancements are extremely intense SEP events.
- High-energy ions and electrons from planetary magnetospheres
  - Jovian electrons
  - Aurora and magnetospheric "substorms"

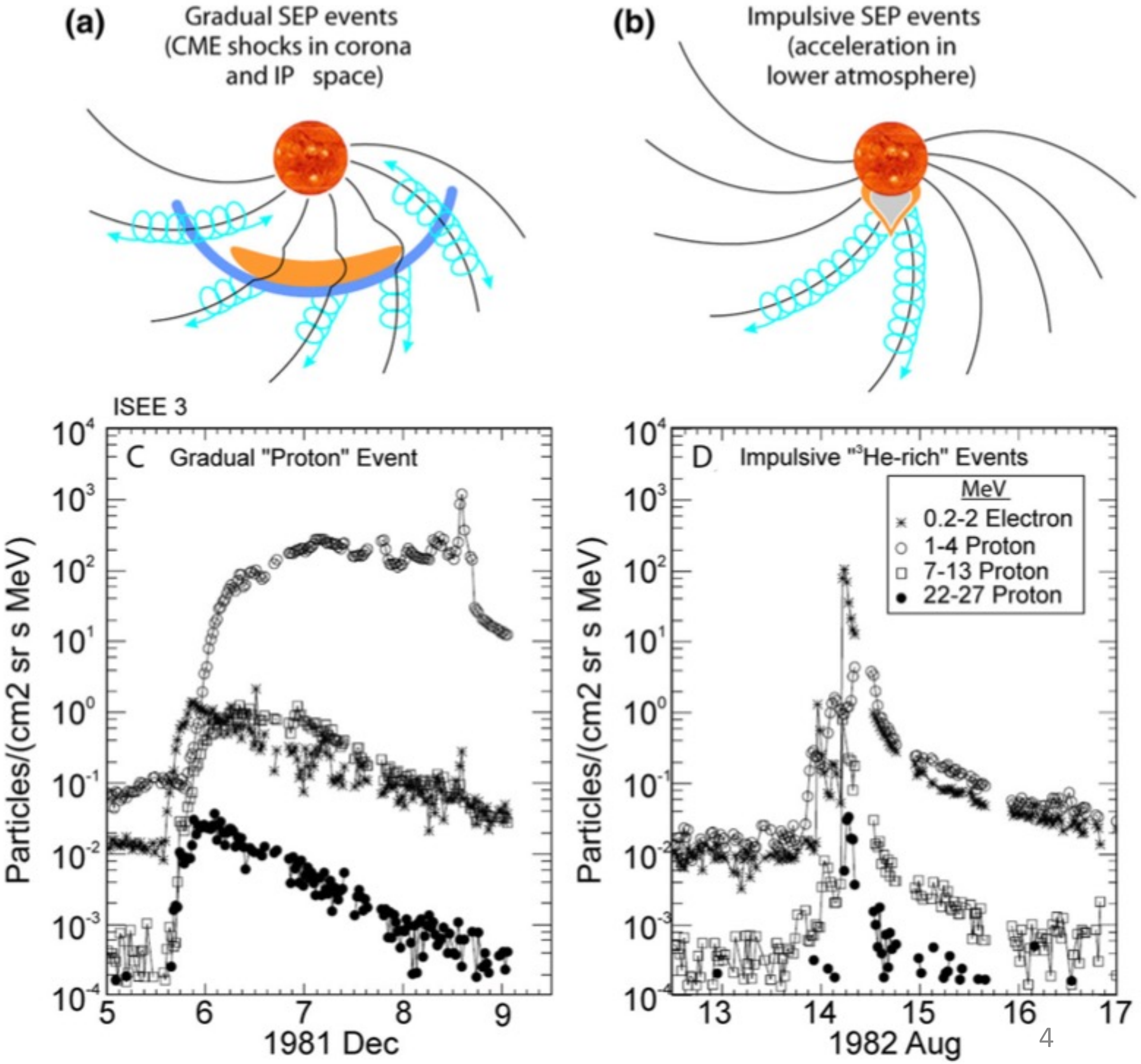


We often characterize energetic particles with its "energy spectrum" which is related to the phase-space distribution function. An example is above from NASAs ACE spacecraft

SEPs are often described with two basic classes

Property	Impulsive	Gradual
Electron/proton	$\sim 10^2$–10^4	$\sim$50–100
^{3}He/^{4}He	$\sim$1	$\sim 4 \times 10^{-4}$
Fe/O	$\sim$1	$\sim$0.1
H/He	$\sim$10	$\sim$100
Q_{Fe}	$\sim$20	$\sim$14
SEP duration	<1–20 h	<1–3 days
Longitude cone	<30°	<100°–200°
Seed particles	Heated Corona	Ambient Corona or SW
Radio type	III	II
X-ray duration	$\sim$10 min–1 h	$\gtrsim$1 h
Coronagraph	N/A	CME
Solar eind	N/A	IP shock
Events/year	$\sim$1000	$\sim$10

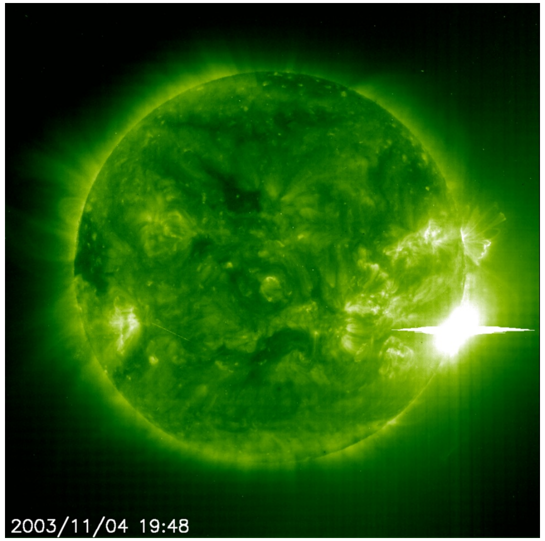
CME-related **flare-related**



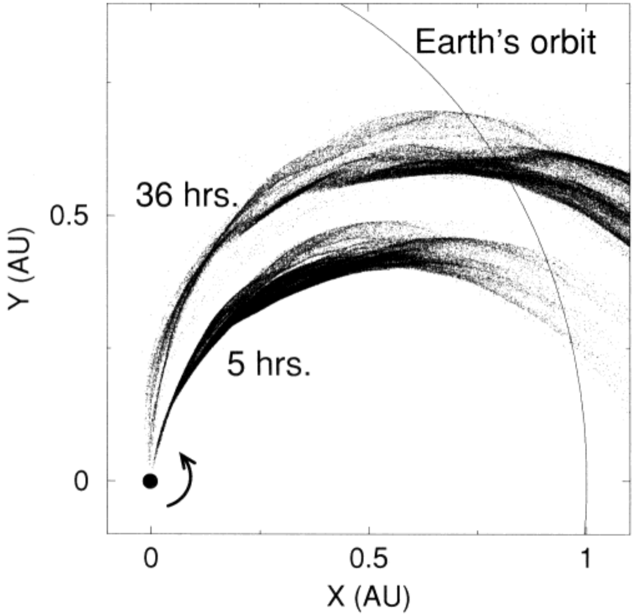
Impulsive Solar-Energetic Particle Events

- Impulsive SEP events seen at 1AU are often characterized by clear velocity dispersion signatures.
  - resembles the "Nike®-swoosh" when plotted as a time vs. MeV/nuc. scatter plot
- That they are impulsive implies that whatever accelerates them does so on a time scale shorter than the transport time scale.
- Ideal for studying charged-particle transport in the interplanetary magnetic field between the Sun and Earth

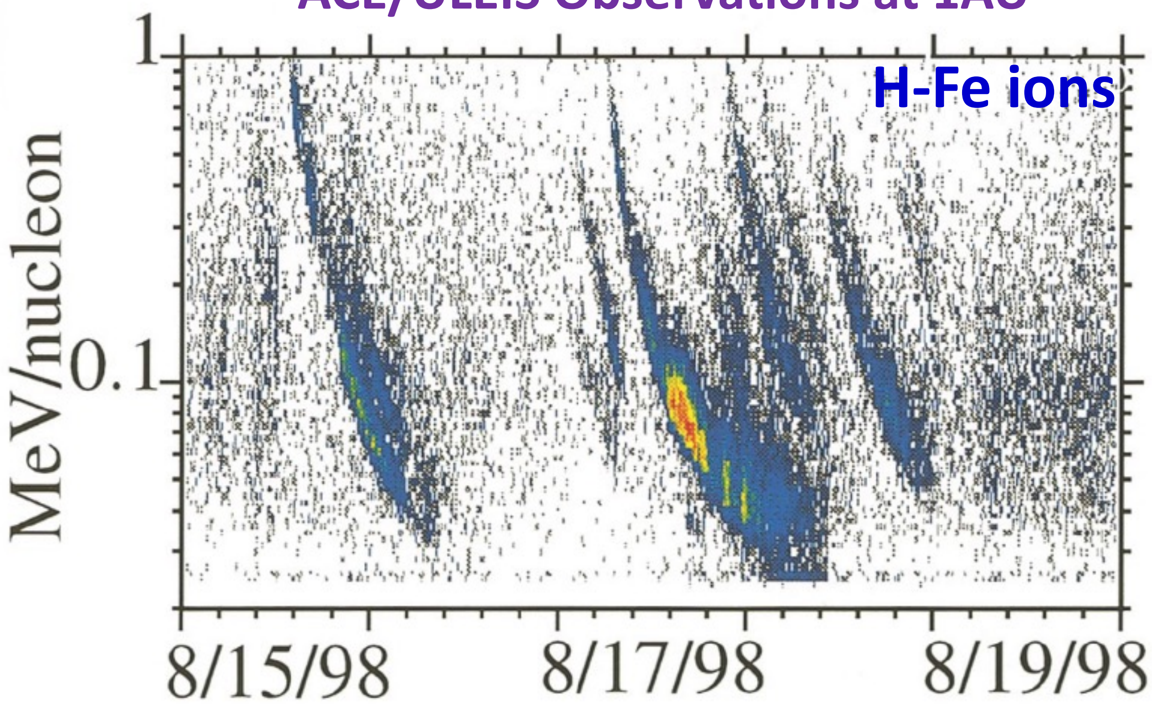
Flare at Sun



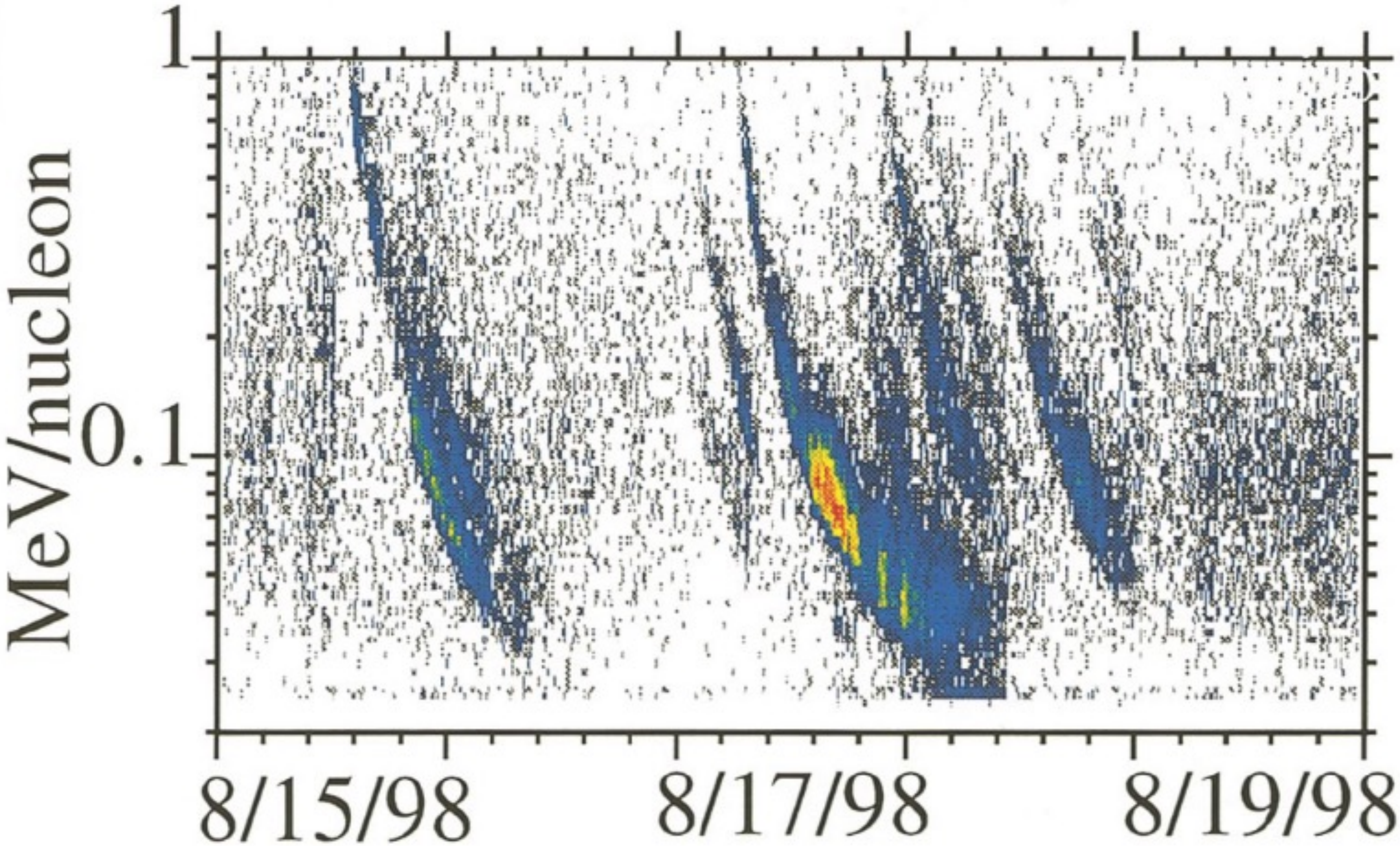
Transport in IMF

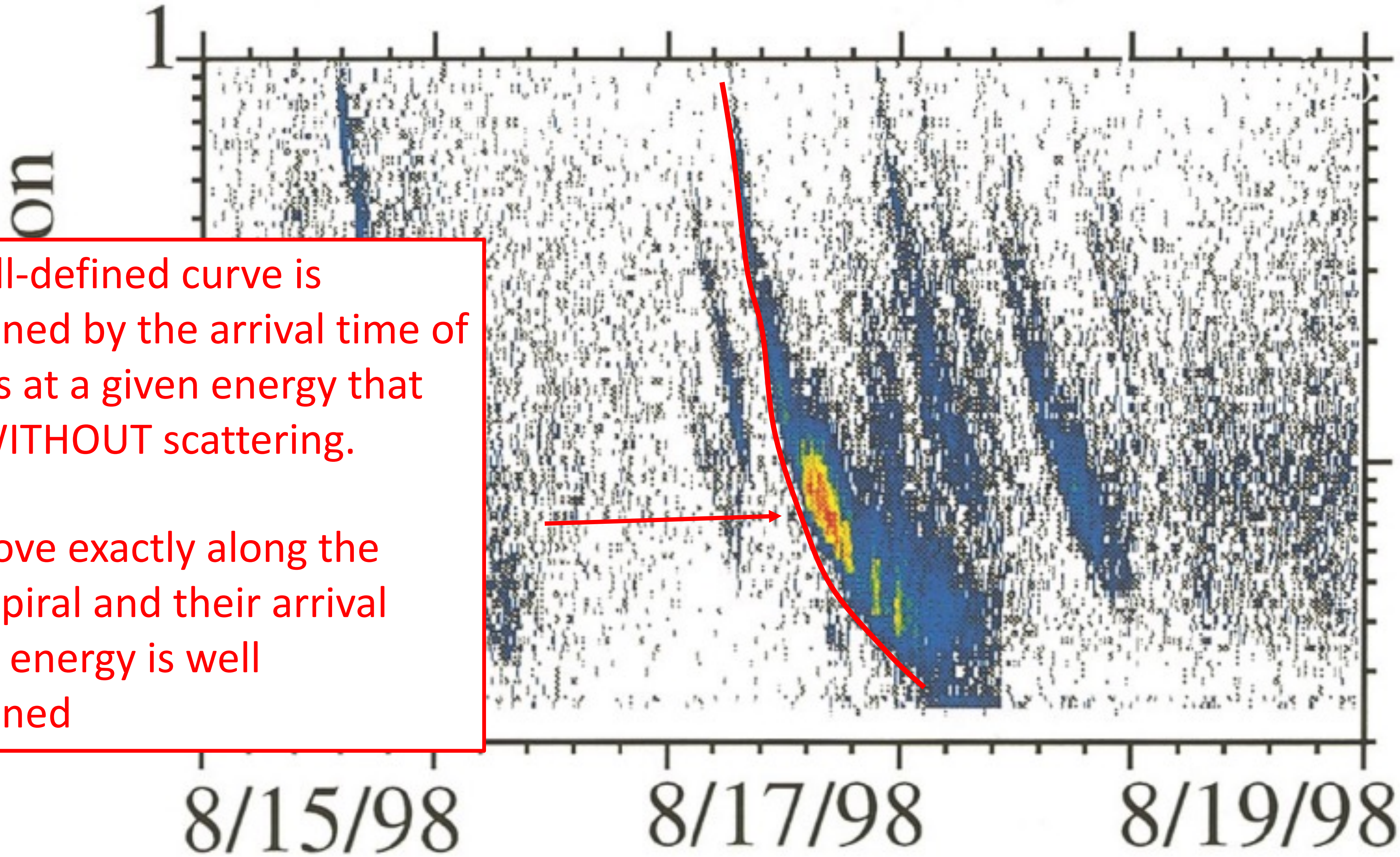


ACE/ULEIS Observations at 1AU



Mason et al., 1999


1
0.1
MeV/nucleon
8/15/98
8/17/98
8/19/98



This well-defined curve is determined by the arrival time of particles at a given energy that move WITHOUT scattering.

They move exactly along the Parker spiral and their arrival time vs. energy is well determined

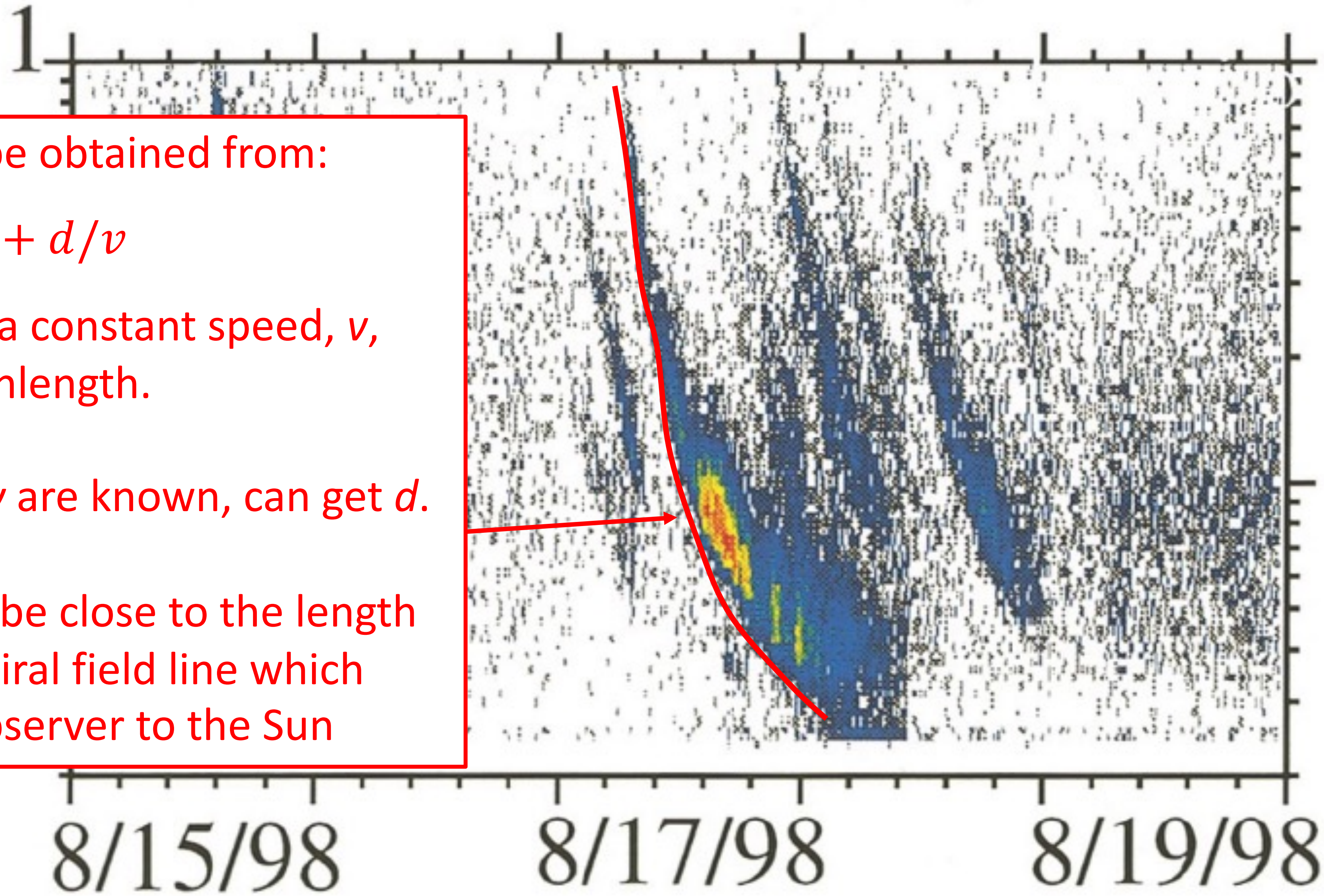
This curve can be obtained from:

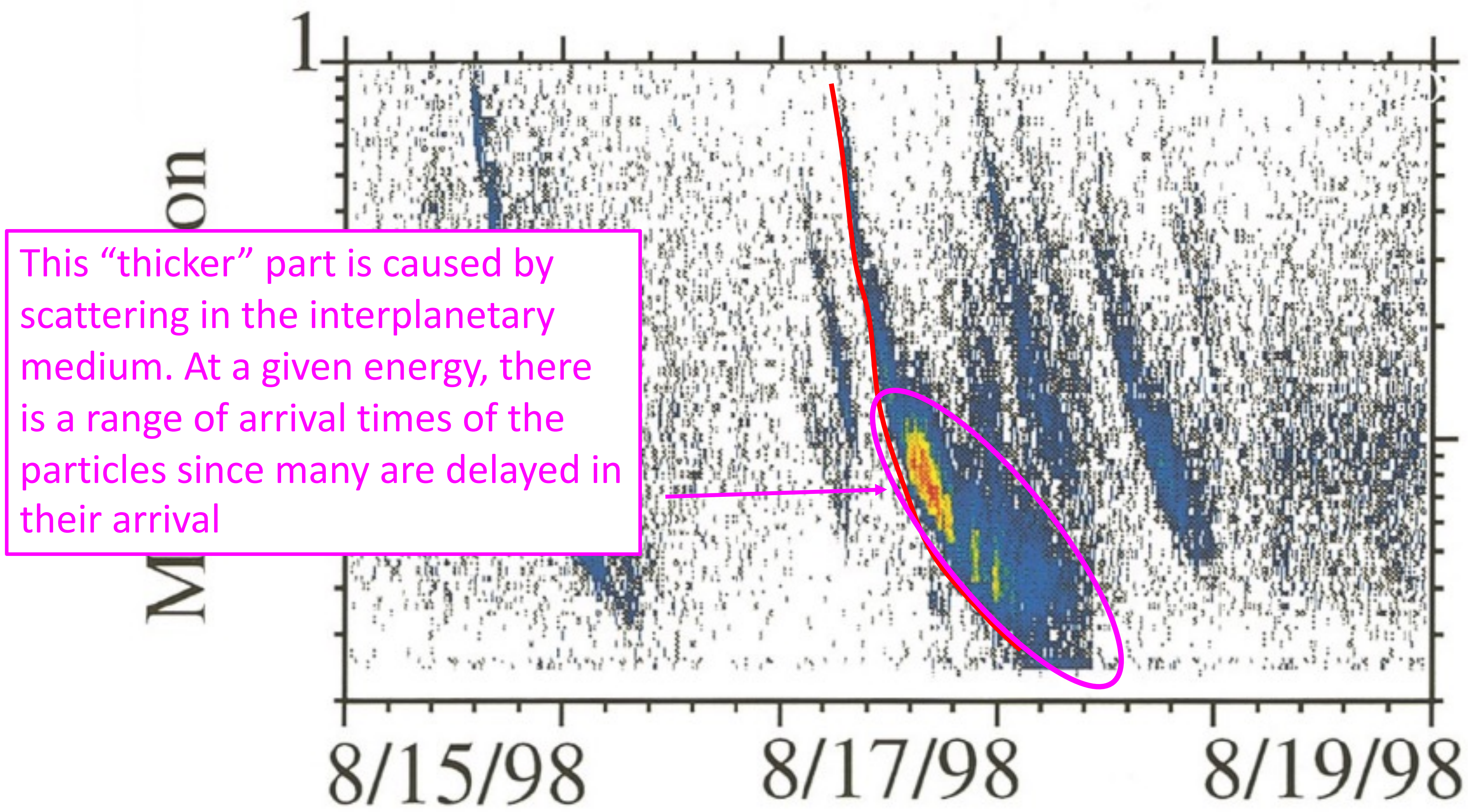
$$t = t_{release} + d/v$$

which assumes a constant speed, *v*, and *d* is the pathlength.

If t, $t_{release}$, and *v* are known, can get *d*.

Ideally, this will be close to the length of the Parker-spiral field line which connects the observer to the Sun

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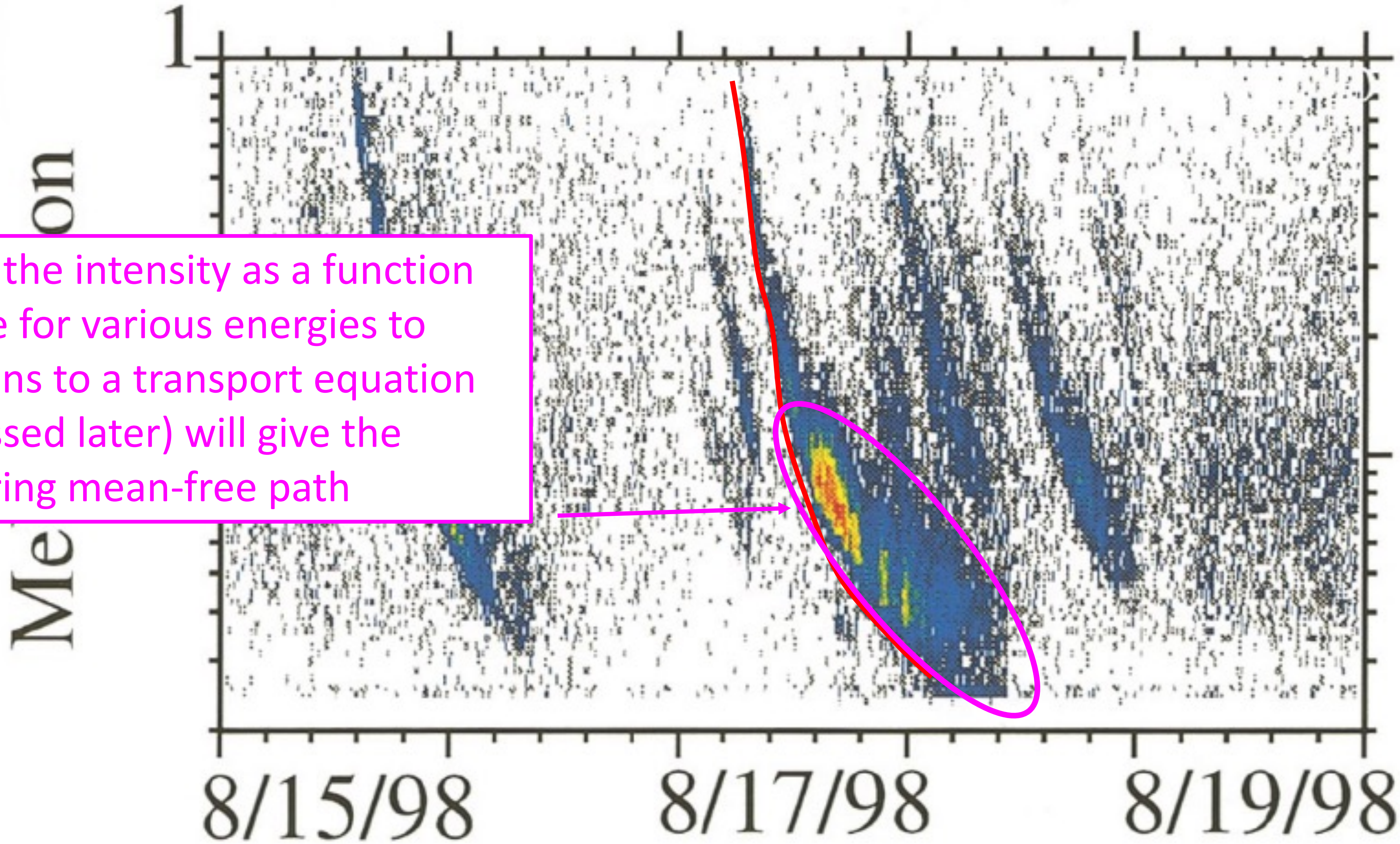
This “thicker” part is caused by scattering in the interplanetary medium. At a given energy, there is a range of arrival times of the particles since many are delayed in their arrival


1
Me
on
Fitting the intensity as a function of time for various energies to solutions to a transport equation (discussed later) will give the scattering mean-free path
8/15/98
8/17/98
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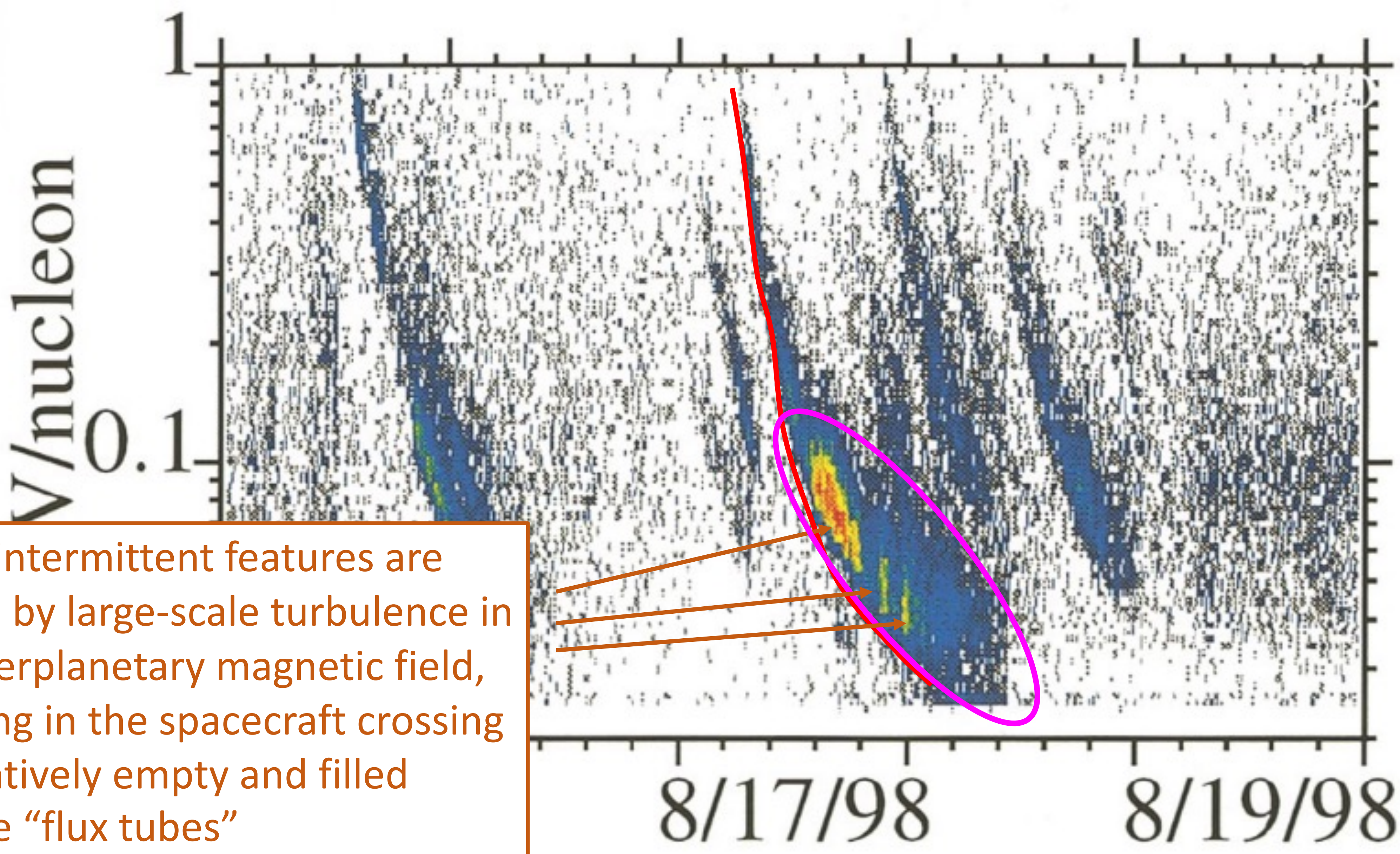
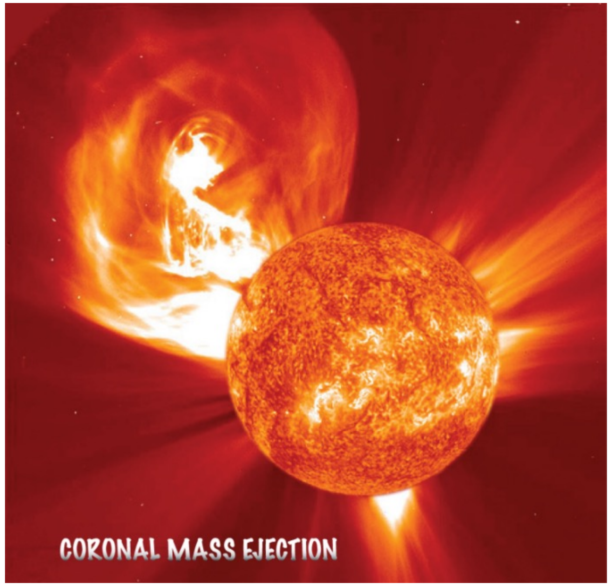
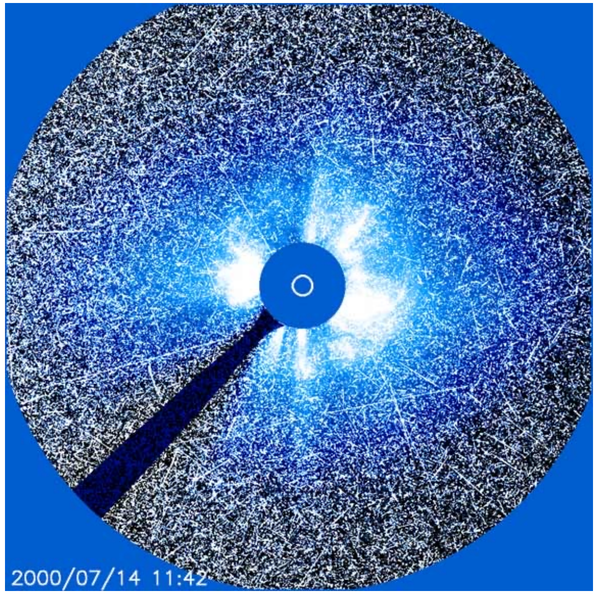


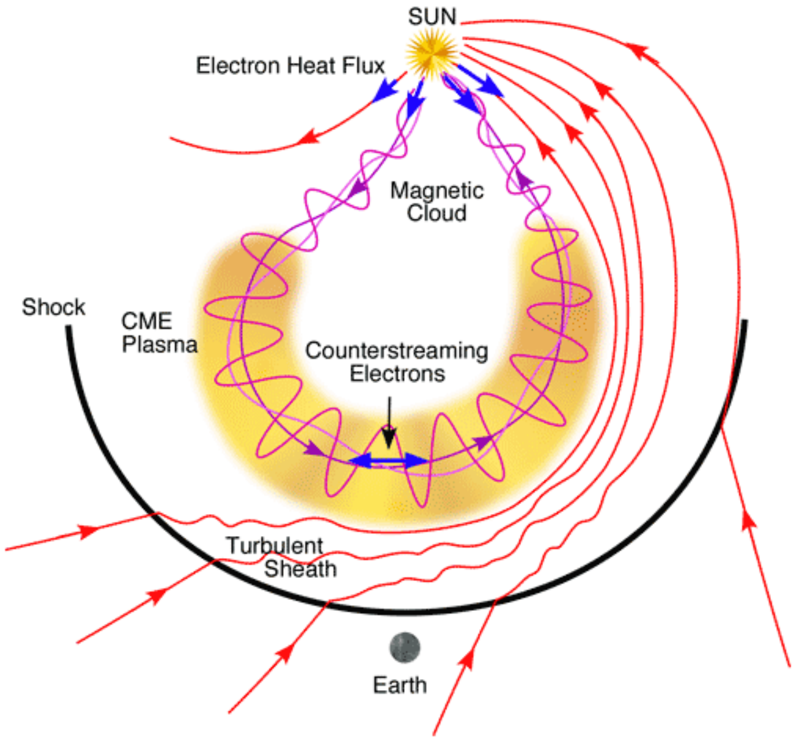
These intermittent features are caused by large-scale turbulence in the interplanetary magnetic field, resulting in the spacecraft crossing alternatively empty and filled particle “flux tubes”

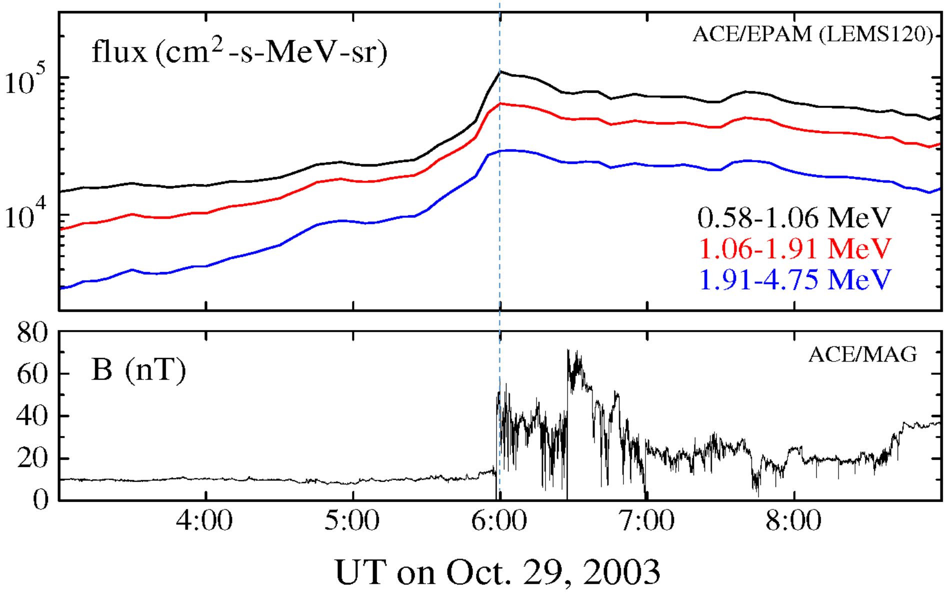
CME-related SEP events

- The largest SEP events are almost always associated with fast CMEs (usually halo CMEs)
- At 1AU, at energies below several MeV, the peak proton intensity is almost always coincident with the passage of the CME shock.

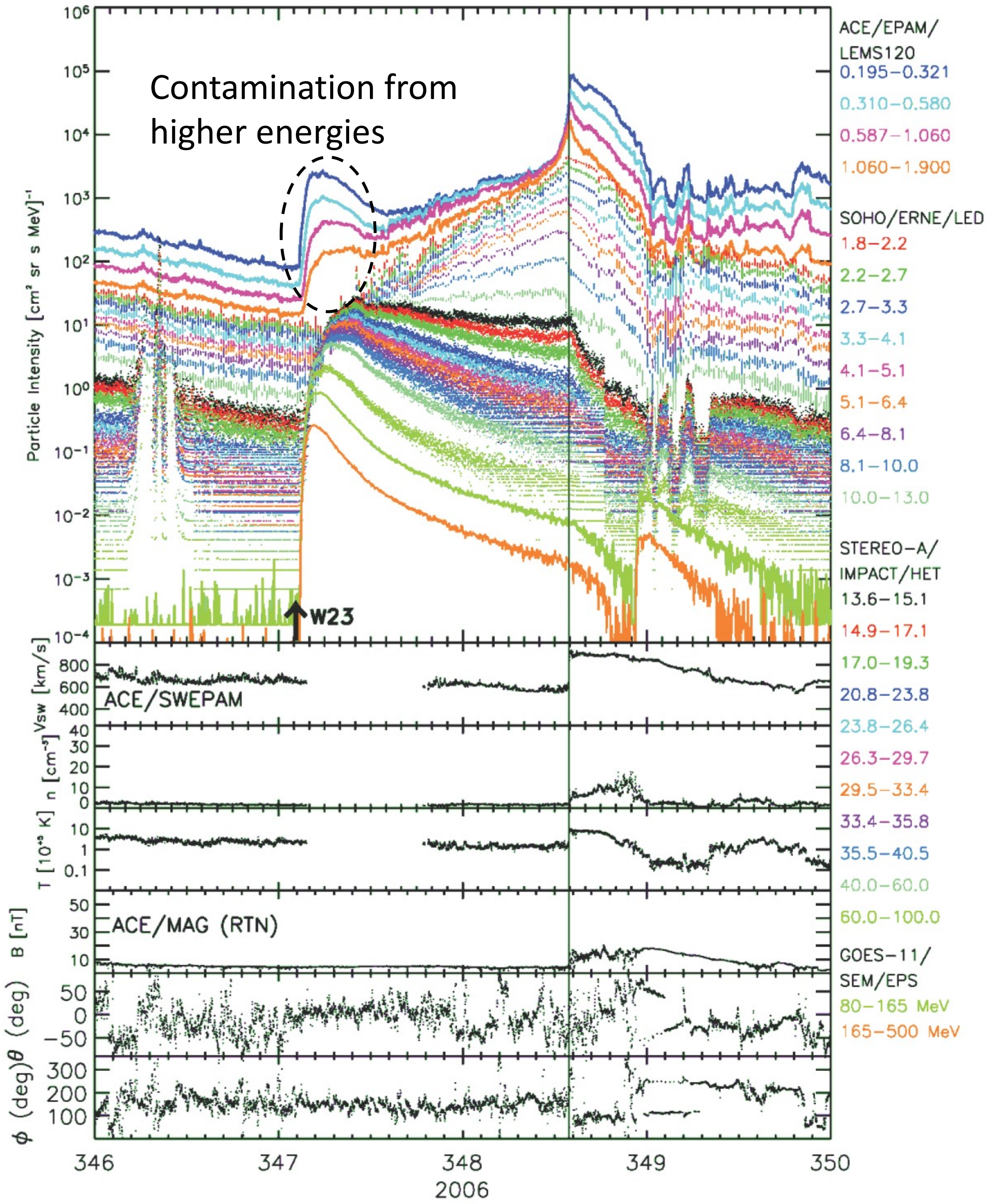




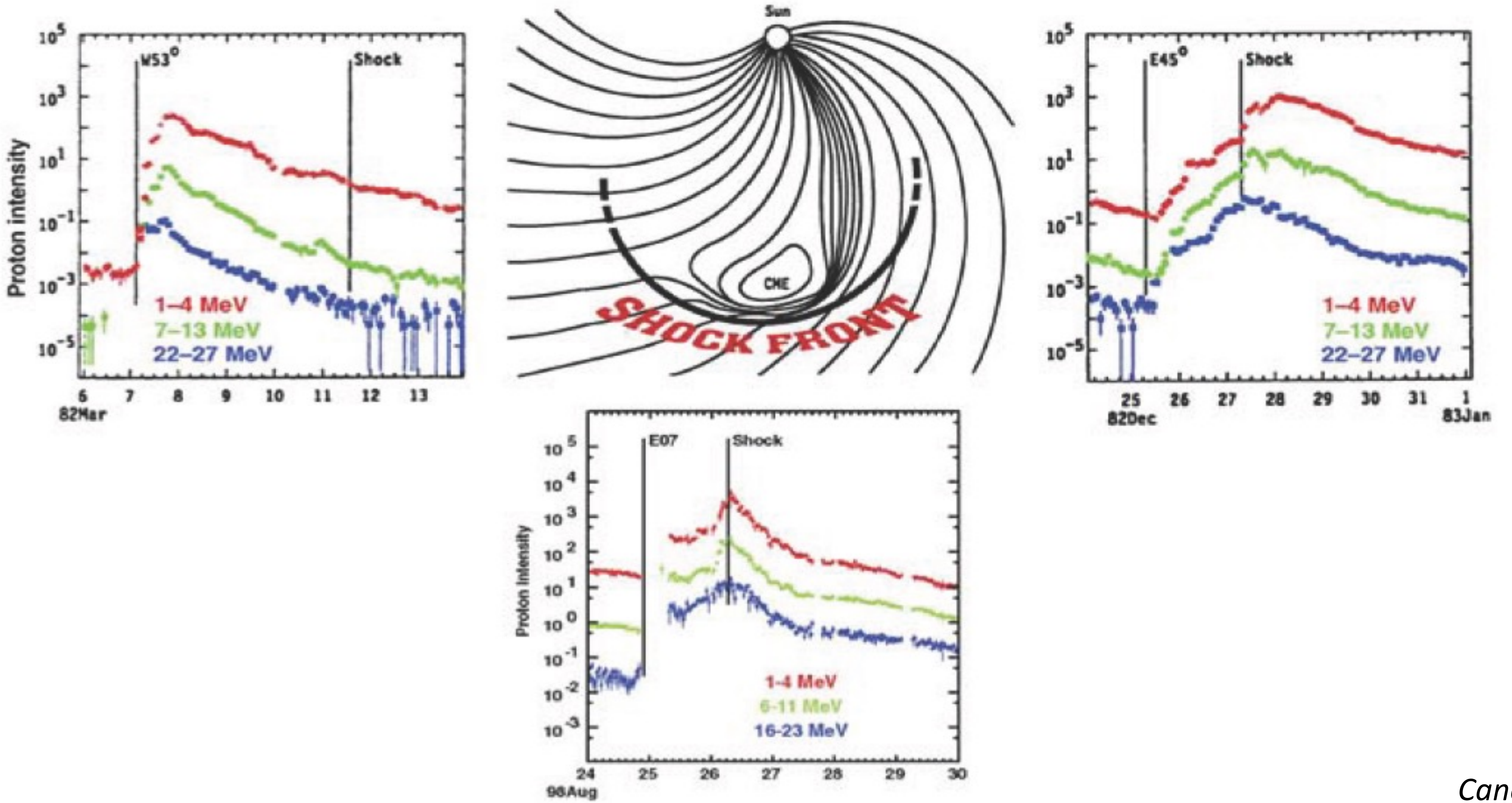




- A typical very large SEP event associated with a fast interplanetary (IP) shock seen by multiple spacecraft (ACE/SOHO/STEREO/GOES) near Earth
- The intensity vs. time depends on energy
- At low energies, the peak intensity is at the IP shock arriving ~1.5 days after the solar event; at high energies, the peak is well before the shock, closer to when the solar event was observed.
- The same shock likely accelerated these particles, but with a rate of acceleration that depends on the location of the shock



CME-related events also depend on the direction the CME is moving relative to the observer



Cane, et al. 1988

The physics of high-energy charged particle transport (= propagation and acceleration).

- Treat as test particles.
- Assume there are no particle-particle collisions
- Equations of motion for any charged particle from the Lorentz force.
  - Need E and B fields.
    - Can get them from MHD simulations, for example, but this not "simple"
    - Can also get them from kinematic models. Easier, perhaps, but how realistic are they?
- Can also average over an ensemble of particles and arrive at very useful equations that describe the collective behavior of the particles, that are generally straightforward to solve numerically on computers
  - Parker transport equation (Parker, 1965)
  - Focused-transport equation (Roelof, 1967; Ruffolo, 1995; Isenberg, 1997; Kota, 2000; Zhang et al., 2009; Droge et al., 2010)

The fundamental equation in ALL transport/acceleration “theories” start with the Lorentz-force on acting on individual charged particle.

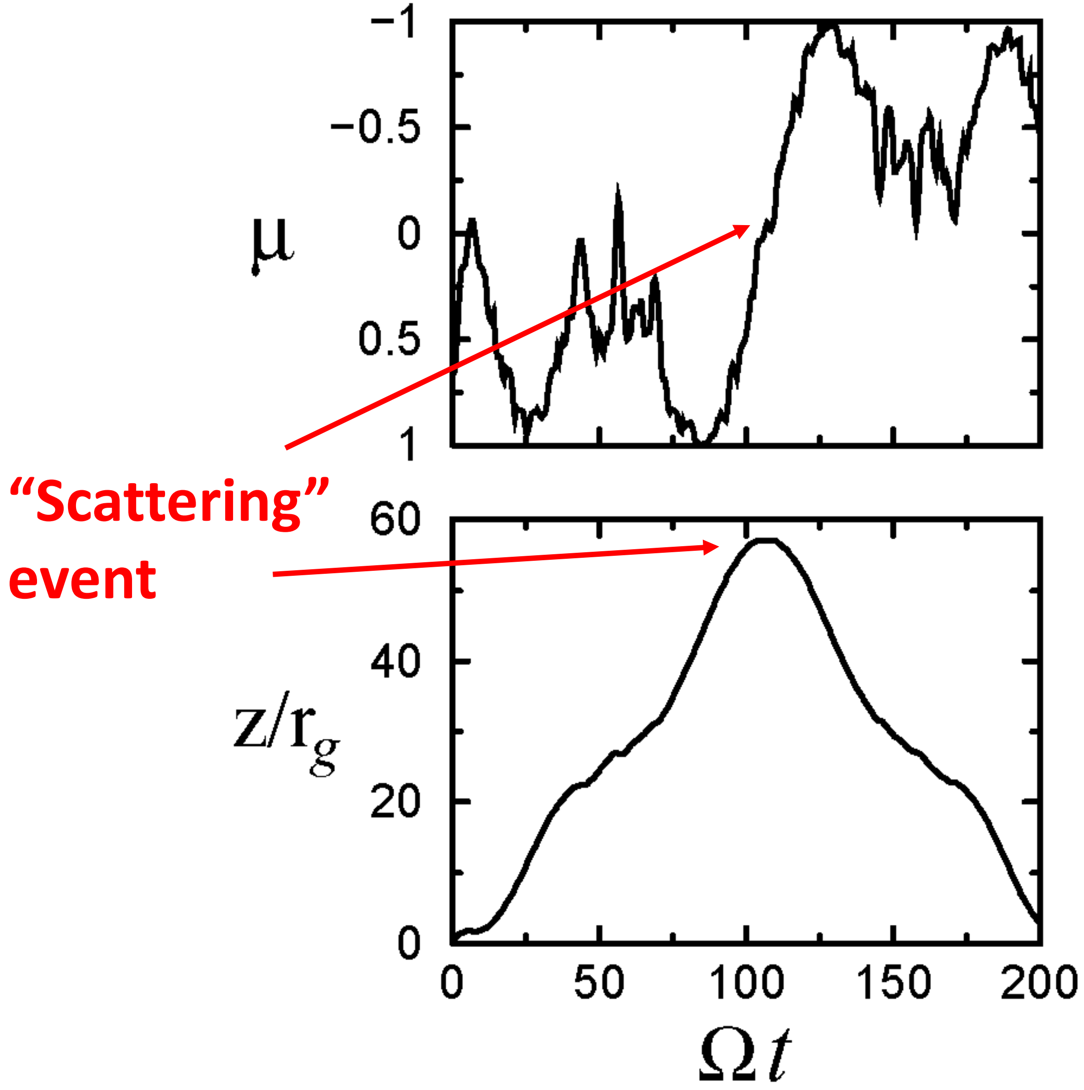
> “Concerning hydrodynamics & magnetohydrodynamics in nature, where no one applies external potentials, ***the dynamics of gases and magnetized plasma is described by the equations of Newton and Maxwell***”
>
> \- Gene Parker, 2009 Summer School in Solar Physics, Sunspot, NM

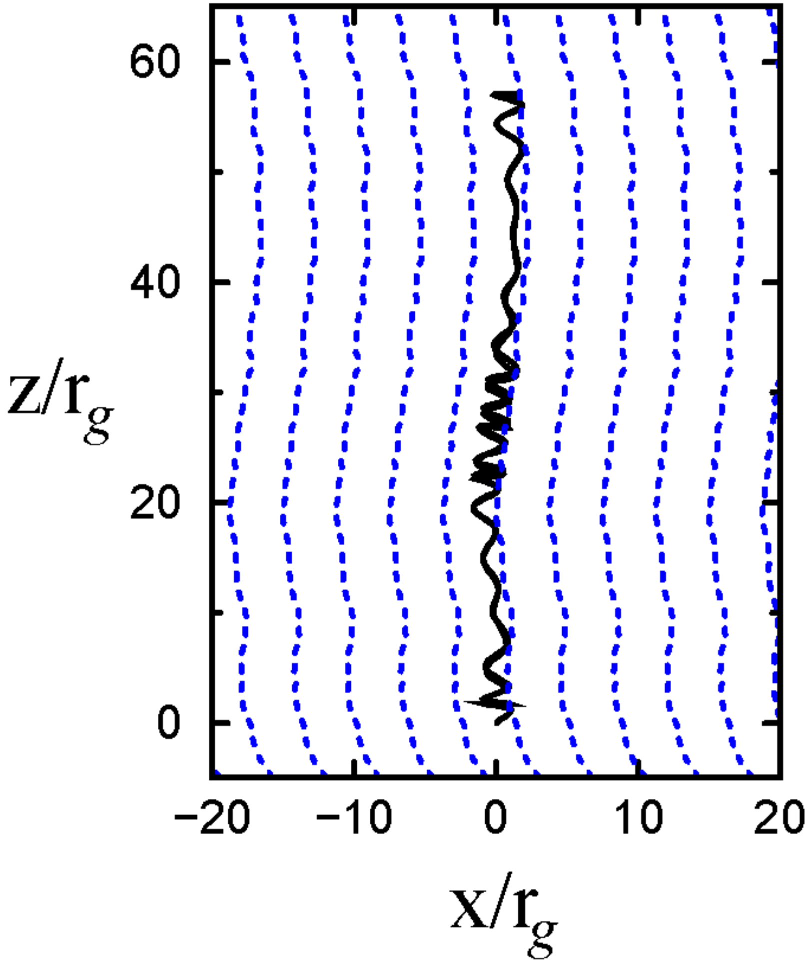
- Lorentz force (cgs, or “Gaussian” units, which I prefer)

$$\frac{d\vec{p}}{dt} = q\vec{E} + \frac{q}{c}\vec{w}\times\vec{B}$$

- Where $\boldsymbol{w} = \boldsymbol{p}/m$ is the particle velocity vector, and $\boldsymbol{p}$ is the momentum. q is the particle’s charge, $\boldsymbol{E}$ and $\boldsymbol{B}$ are the electric and magnetic fields, respectively.
- Other forces are generally small, but can be added as needed (e.g. gravity, radiation pressure, etc.)

A charged particle moving in a turbulent magnetic field (numerical integration)





Note that this particle seems “stuck” on a magnetic field line.

It is!

This is artificial and a consequence of the assumed reduced dimensionality of the magnetic field. Be careful of this!

Under the hood: The underlying physics of particle transport

Pitch-angle , spatial diffusion, and the quasi-linear th

- We assume charged particles undergo pitch-angle diffusion and their distribut

$$\frac{df(\mu,z,t)}{dt} = -w\mu\frac{\partial f(\mu,z,t)}{\partial z} + \frac{\partial}{\partial\mu}\left(D_{\mu\mu}(\mu)\frac{\partial f(\mu,z,t)}{\partial\mu}\right)$$

This i
Boltz

- $D_{\mu\mu}$ is the pitch-angle diffusion coefficient, μ is the pitch cosine. w is the partic the direction along the mean magnetic field.
- By assuming small anisotropy, this can be written as a spatial diffusion equatio

$$\frac{df(z,t)}{dt} = \frac{\partial}{\partial z}\left(\kappa_{\parallel}\frac{\partial f(z,t)}{\partial z}\right)$$

Jokipii, (1966, 1969)
Hasselmann & Wibberenz
Earl, (1974)
Luhmann, (1978)

- Where $\kappa_{\parallel}$, is related to the $D_{\mu\mu}$ by

$$\kappa_{\parallel}(w) = \frac{w^2}{4}\int_0^1 \frac{(1-\mu^2)^2}{D_{\mu\mu}} d\mu$$

Mean-free path $\lambda_{\parallel}$

Pitch-angle , spatial diffusion, and the quasi-linear theory (cont.)

- The "scattering" is caused by turbulent magnetic fields. $D_{\mu\mu}$ is obtained by integrating the equations of motion of particles moving within these fields (*Jokipii*, 1966). For a mean magnetic

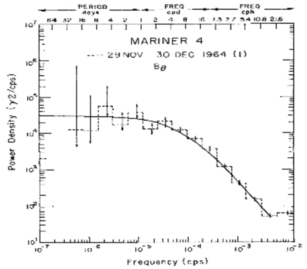
$$\frac{dp_z}{dt} = \frac{q}{c}\left(v_x\delta B_y - v_y\delta B_x\right)$$

- Solved by inserting the zeroth order fluctuating field) into the above, dro quantities, integrating over time, ave realizations. One obtains the well-kn

$$D_{\mu\mu} = \lim_{\Delta t\to\infty}\frac{\langle(\Delta\mu)^2\rangle}{2\Delta t} = \frac{\pi}{4}\Omega_0$$

Pitch-angle , spatial diffusion, and the quasi-linear theory (cont.)

Jokipii & Coleman, JGR, 1968



Interplanetary turbulence

- Assuming a power spectrum of the form:

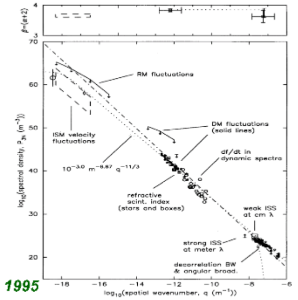
$$P(k) = \frac{C}{1+(kL_c)^{5/3}} \qquad \sigma^2 = \int_0^{\infty} P(k)dk$$

- L_c is the turbulence correlation length. We obtain

$$\kappa_{\parallel} = \frac{3v^2}{20L_c\Omega_0^2}\left(\frac{B_0}{\sigma}\right)^2 \csc(3\pi/5)\left[1+\frac{72}{7}\left(\frac{\Omega_0 L_c}{v}\right)^{5/3}\right]$$

- Can also solve for cross-field transport

$$\kappa_{\perp} = \frac{5vL_c}{12}\left(\frac{\sigma}{B_0}\right)^2 \sin(3\pi/5)$$



Interstellar turbulence

Armstrong et al, 1995

In addition to diffusion, there are other important collective effects on charged particles: they include

- Advection

$$\vec{U}\cdot\nabla f$$

(arises because the "scattering centers" are moving with the bulk plasma flow)

- Energy Change

$$\frac{p}{3}\nabla\cdot\vec{U}\frac{\partial f}{\partial p}$$

(arises because of scattering in converging or diverging flows)

- Note that the energy-char In fact, the electric field is
- Starting with the work do

$$\frac{d}{dt}\left(\frac{p^2}{2m}\right) = -q\vec{w}\cdot\vec{E} = \frac{q}{c}\vec{w}\cdot\left(\vec{U}\times\vec{B}\right)$$

- Then, by using a vector identity, and averaging over an isotropic distribution of particles, it follows (c.f. Jokipii, AIP Conf. Proc., 2012)

$$\frac{dp'}{dt} = -\frac{p'}{3}\nabla\cdot\vec{U}$$

- Which appears in the energy-change term, where p' is the momentum in the frame of reference moving with ***U***.

The Parker Transport Equation (Parker, 1965):

$$\frac{\partial f}{\partial t} = -V_{w,i}\frac{\partial f}{\partial x_i} + \frac{\partial}{\partial x_i}\kappa_{ij}^{(S)}\frac{\partial f}{\partial x_j} - V_{D,i}\frac{\partial f}{\partial x_i} + \frac{1}{3}\frac{\partial V_{w,i}}{\partial x_i}\frac{\partial f}{\partial \ln p} + Q$$

advection ***diffusion*** ***drift*** ***energy change*** ***source***

Where the drift velocity due to the large scale curvature and gradient of the average magnetic field is:

$$\mathbf{V_d} = \frac{pcw}{3q}\nabla \times \left[\frac{\mathbf{B}}{B^2}\right]$$

And the symmetric part of the diffusion tensor is:

$$\kappa_{ij}^{(S)} = \kappa_\perp \delta_{ij} + (\kappa_\parallel - \kappa_\perp)\frac{B_i B_j}{B^2}$$

The Parker Transport Equation is valid whenever the anisotropy is small (as observed for GCRs). It is widely used and remarkably general.

Solving the Parker equation for SEP studies

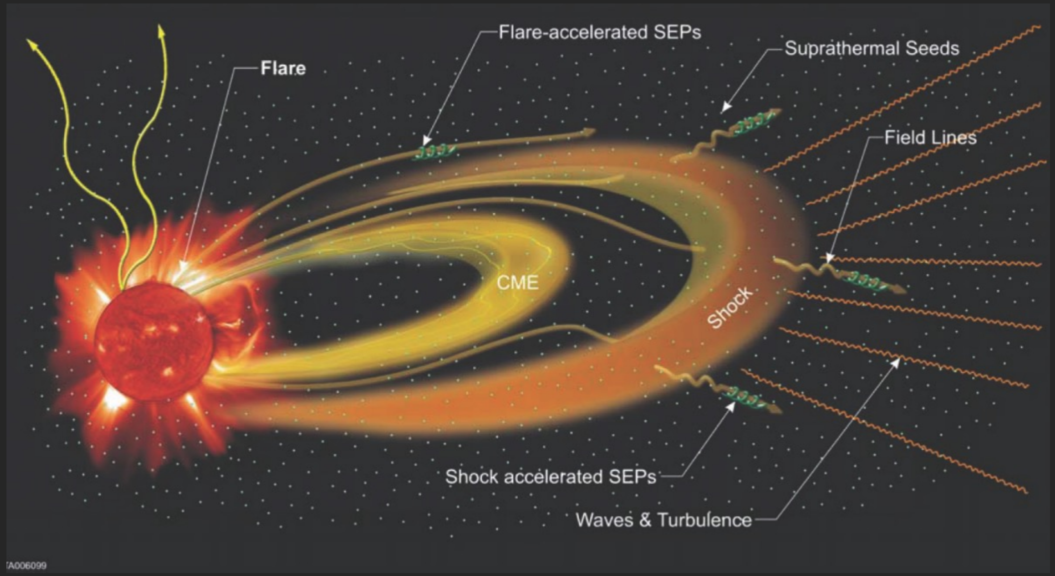
- Generally closed-form analytic solutions to the Parker equation only exist in very simplified situations.
- There are two popular ways to solve it numerically

Finite-difference integration

- Advantage: “perfect” statistics.
- Disadvantage: numerical stability, difficulty solving in 3D (cross-derivatives).

Stochastic Integration

- Advantage: easy to implement on a computer. Perfectly stable.
- Disadvantage: statistics



Finite-difference approach

- Solve the Parker equation directly by writing derivatives as:

$$\frac{\partial f(x,p,t)}{\partial t} \rightarrow \frac{f_{i,j}^{n+1} - f_{i,j}^{n}}{\Delta t} \qquad \frac{\partial f(x,p,t)}{\partial x} \rightarrow \frac{f_{i+1,j}^{n} - f_{i-1,j}^{n}}{2\Delta x}$$

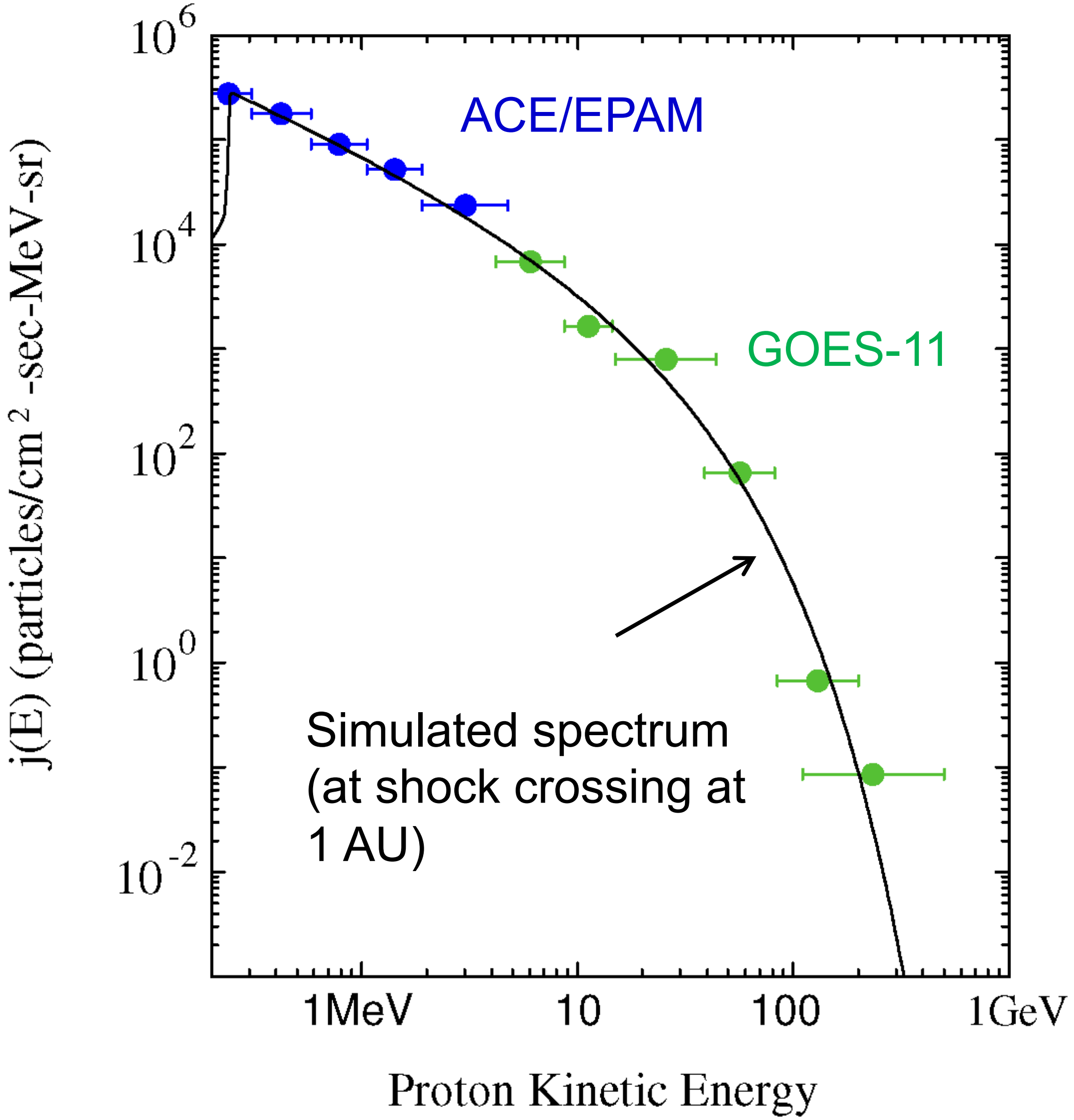
"centered" differencing

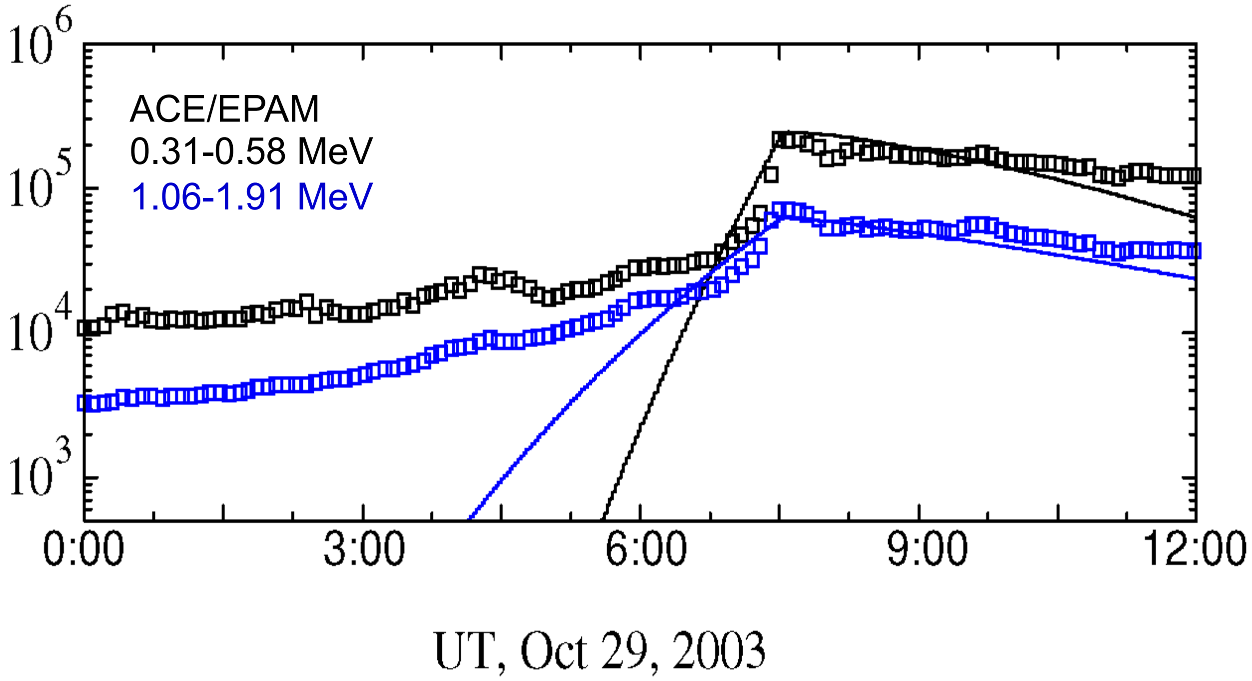
"upwind" differencing

$$\frac{\partial f(x,p,t)}{\partial p} \rightarrow \frac{f_{i,j+1}^{n} - f_{i,j}^{n}}{\Delta t} \qquad \frac{\partial^2 f(x,p,t)}{\partial x^2} \rightarrow \frac{f_{i+1,j}^{n} - 2f_{i}^{n} + f_{i-1,j}^{n}}{\Delta x^2}$$

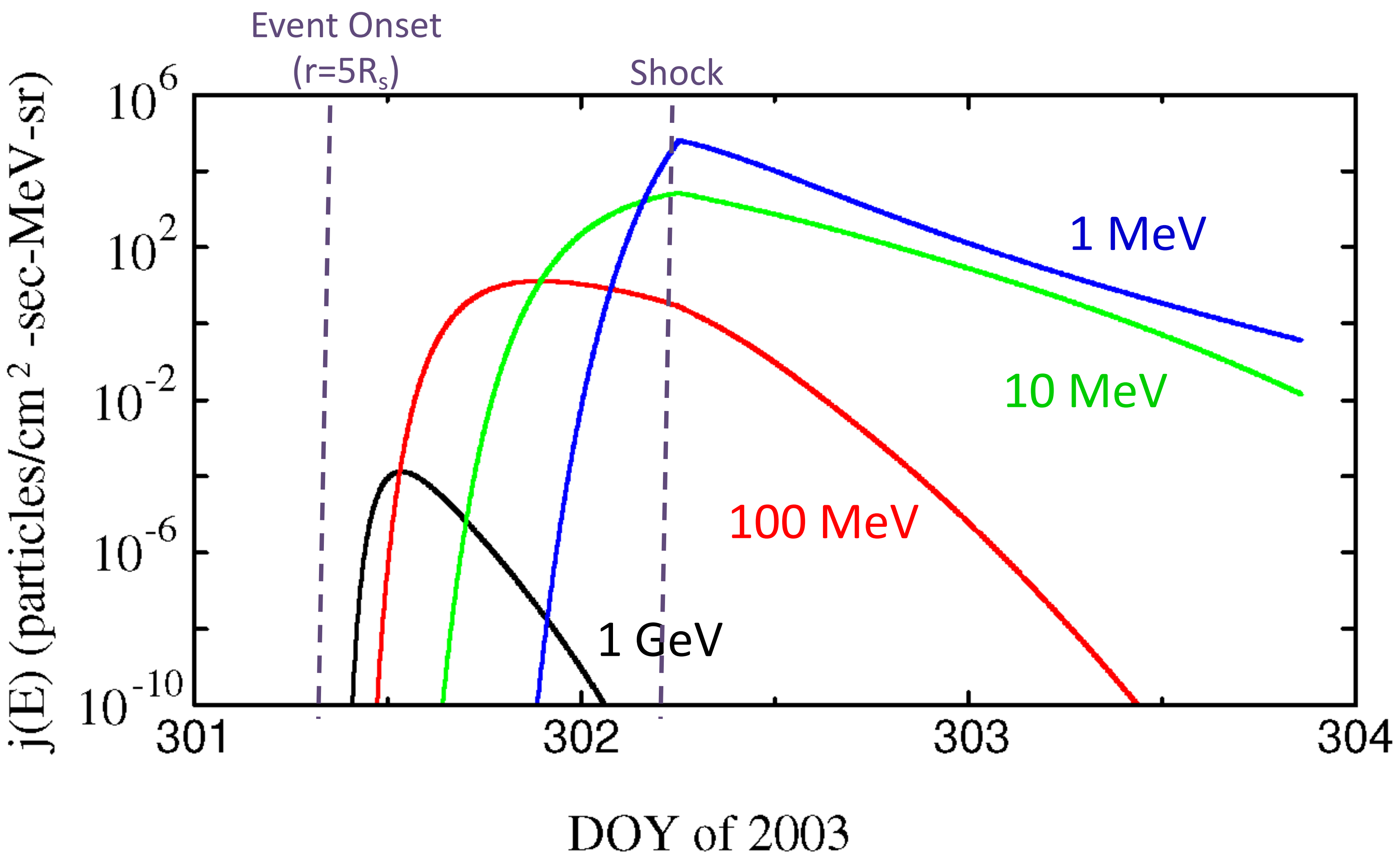
- There are many ways to do this. If the spatial/momentum derivatives are evaluated at the "n+1" time step, then this is called "implicit" (stable). Explicit is the case shown above where these derivatives are evaluated at the "n" time step and is generally unstable and requires very small time steps.
- Choosing upwind/downwind differencing for the momentum change depends on the divergence of the plasma velocity (whether it is acceleration or deceleration).
- The resulting finite difference equation, after being written out in its entirety, is known as the "tridiagonal matrix method"
- There is usually a limit on the time step for stability.
- This approach is generally straightforward for 1-D (spatial) problems, but is more difficult in 2 or 3 dimensions. Cross-derivatives are tricky!
- There are a number discussions on the finite-difference approach to solving equations in textbooks.

This method was used for SEPs accelerated by spherical shock propagating from the Sun by Giacalone, ApJ, 2015.





SEP intensities vs. time at 1 AU



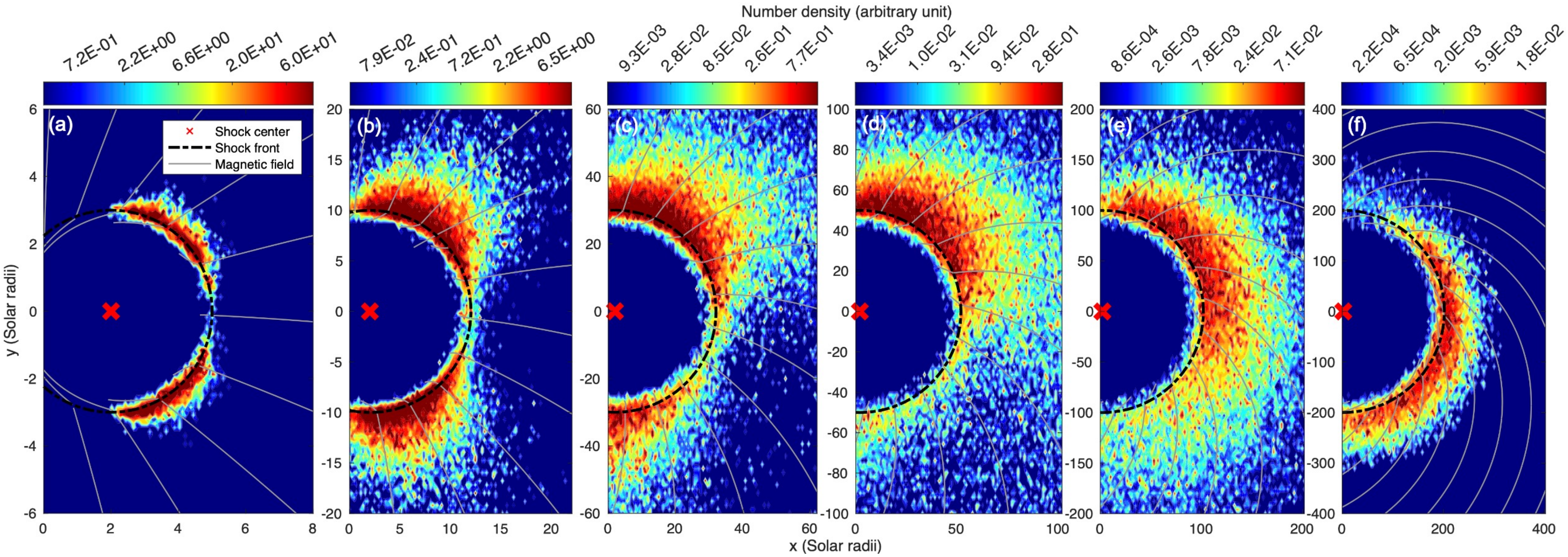
Stochastic integration approach

- In this method, elements of the distribution function are treated like "particles" with each particles position and momentum determined by stochastic equations. A simple example, for 1D diffusion/advection, is:

$$x^{n+1} = x^n + U_x \Delta t + \xi\sqrt{2\kappa\Delta t} \qquad p^{n+1} = p^n\left(1 - \frac{\Delta t}{3}\nabla\cdot\vec{U}\right)$$

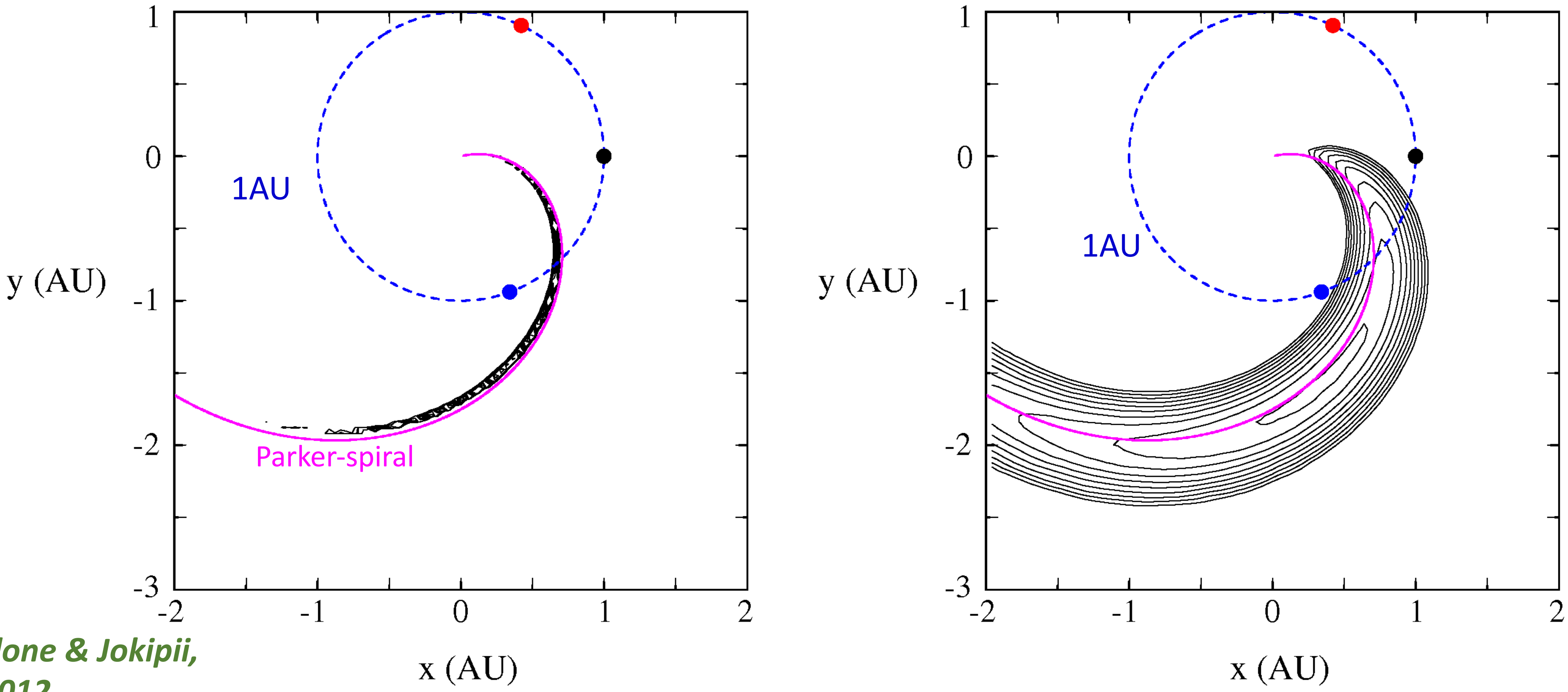
- Similar equations can be written in 2D and 3D.
- This method is easy to implement on a computer, and with the great speed of modern computers, it is not overly computationally expensive.
- This method is used commonly in studies of cosmic rays as well.
- One must not think of these as real particles, however, they are elements of the distribution function!
  - For instance, $(x^{n+1}-x^n)/\Delta t$ is NOT the particle speed!

This method was used to study particle acceleration at a shock in which the local magnetic-field / unit-shock-normal vary along the shock. This is the case of a CME-shock



Work by Xiaohang Chen, University of Arizona

This method was also used to study Particle transport from impulsive solar flares



Giacalone & Jokipii, ApJ, 2012

Numerical simulation of the distribution function (contours) of SEPs associated with a <u>solar flare</u>

The two different plots are for two different values of cross-field diffusion

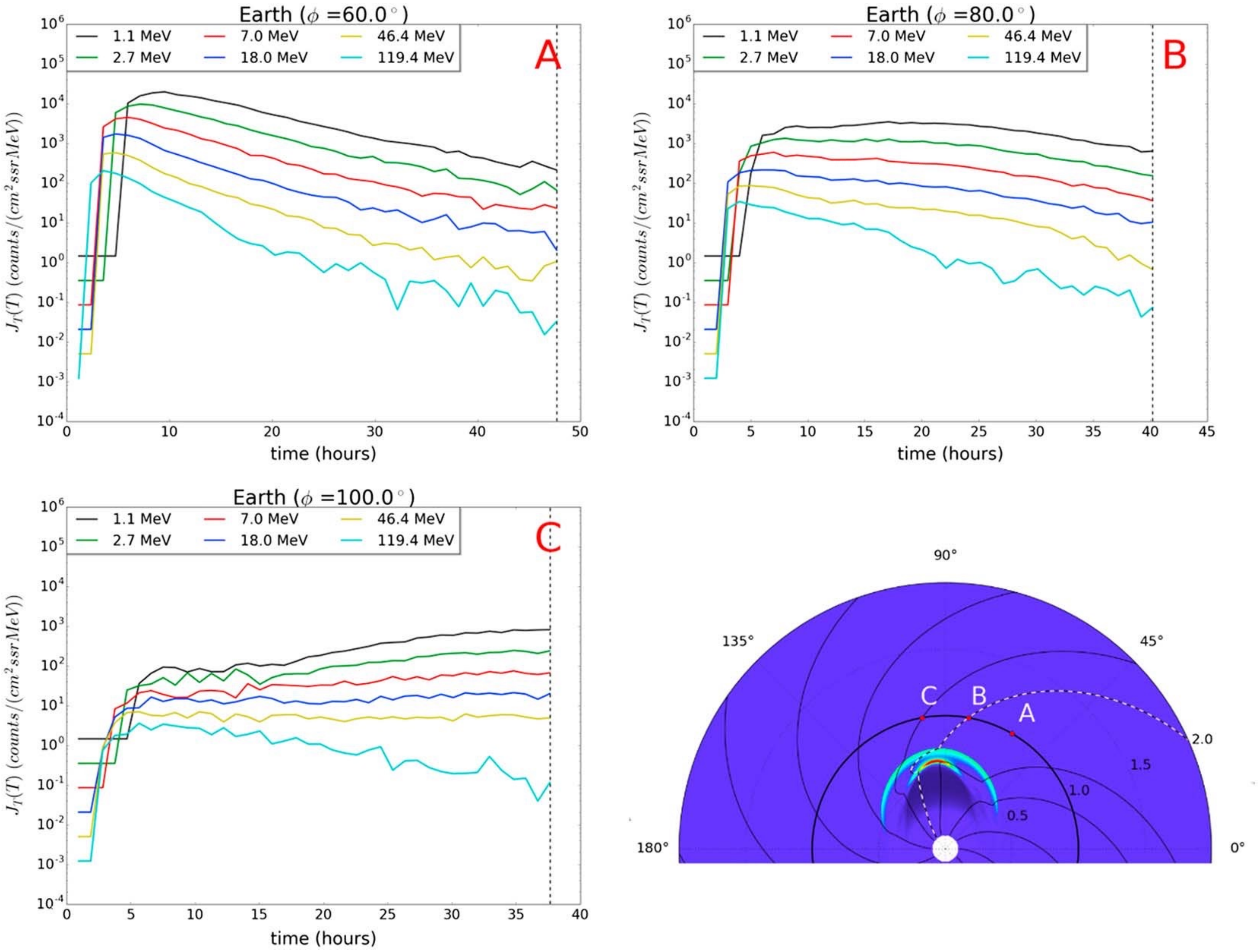
The iPATH model

(Hu et al, 2017; Li et al 2021)

A combination of Diffusive Shock Acceleration (DSA) and Focused transport equation

The spectrum at the shock is determined by assuming the results of DSA theory (discussed a bit later), while the focused transport is used to get distribution far from the shock

Time-intensity profiles of SEPs from this model



Direct orbit-integration approach

- It is also straightforward to numerically solve the force equation itself, and solve for the trajectories of a large number of individual particles.

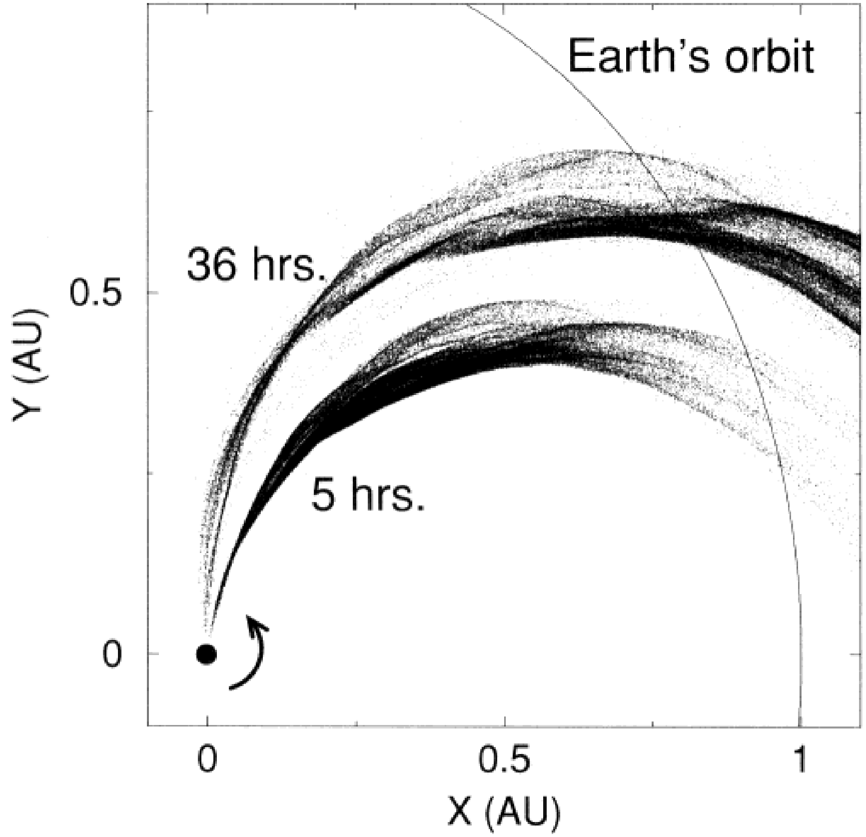
$$\frac{d\vec{p}}{dt} = q\vec{E} + \frac{q}{c}\vec{w}\times\vec{B}$$

Solved numerically with various common approaches: leap-frog, Runga-Kutta, Burlish-Stoer, etc. These can be found in numerical methods textbooks

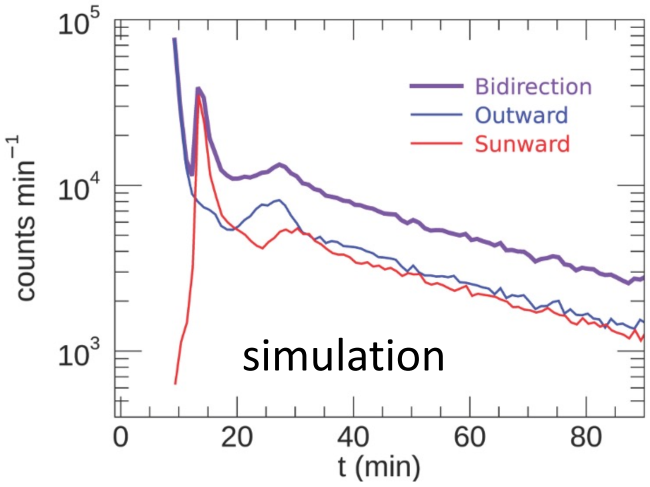
- This approach contains the most physics
- Modern computers are fast enough that this is a reasonable approach.
- Need to know the electric and magnetic fields!
  - Can get them from MHD simulations
    - Need to be careful with grid resolution!
  - Can get them from kinematic models and even include turbulence

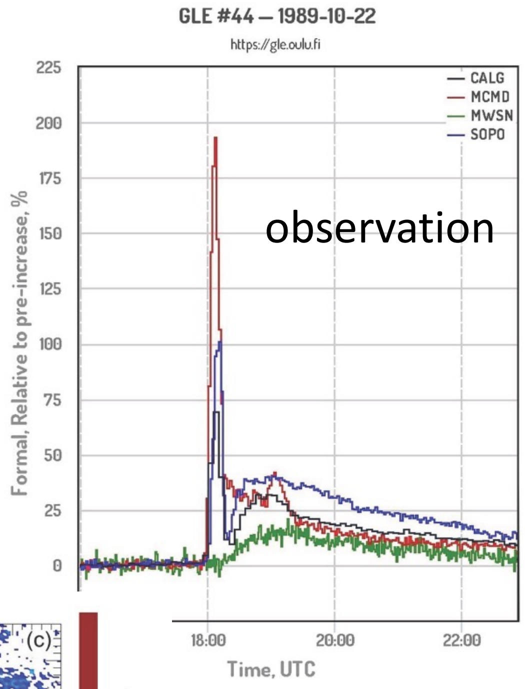
Some examples of studies using this approach

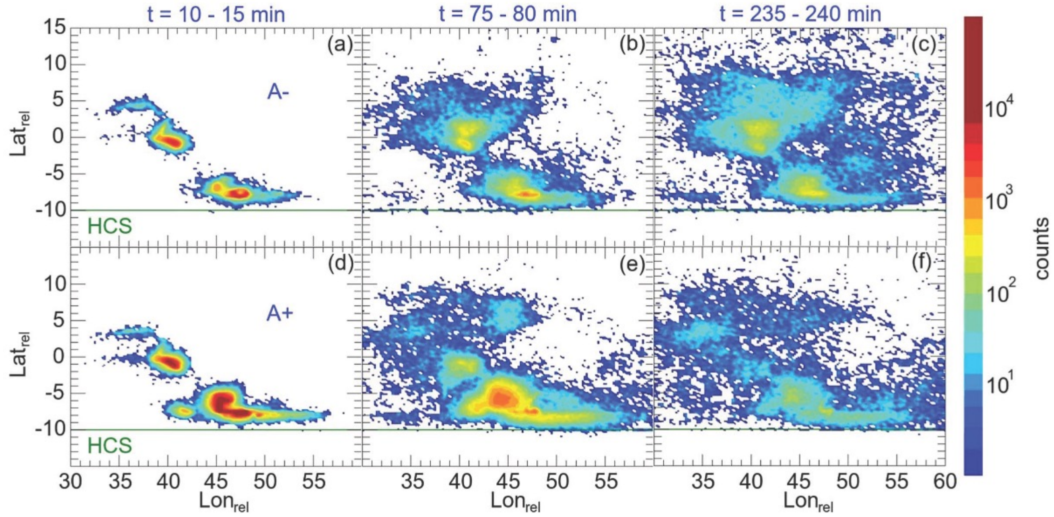
SEPs from Impulsive Flares moving in meandering magnetic fields (*Giacalone, et al., ApJ, 2000*)



Very high-energy SEPs leading to "Gound-Level Enhancements" seen by Neutron Monitors on Earth (*Moradi & Giacalone, ApJ, 2022*)





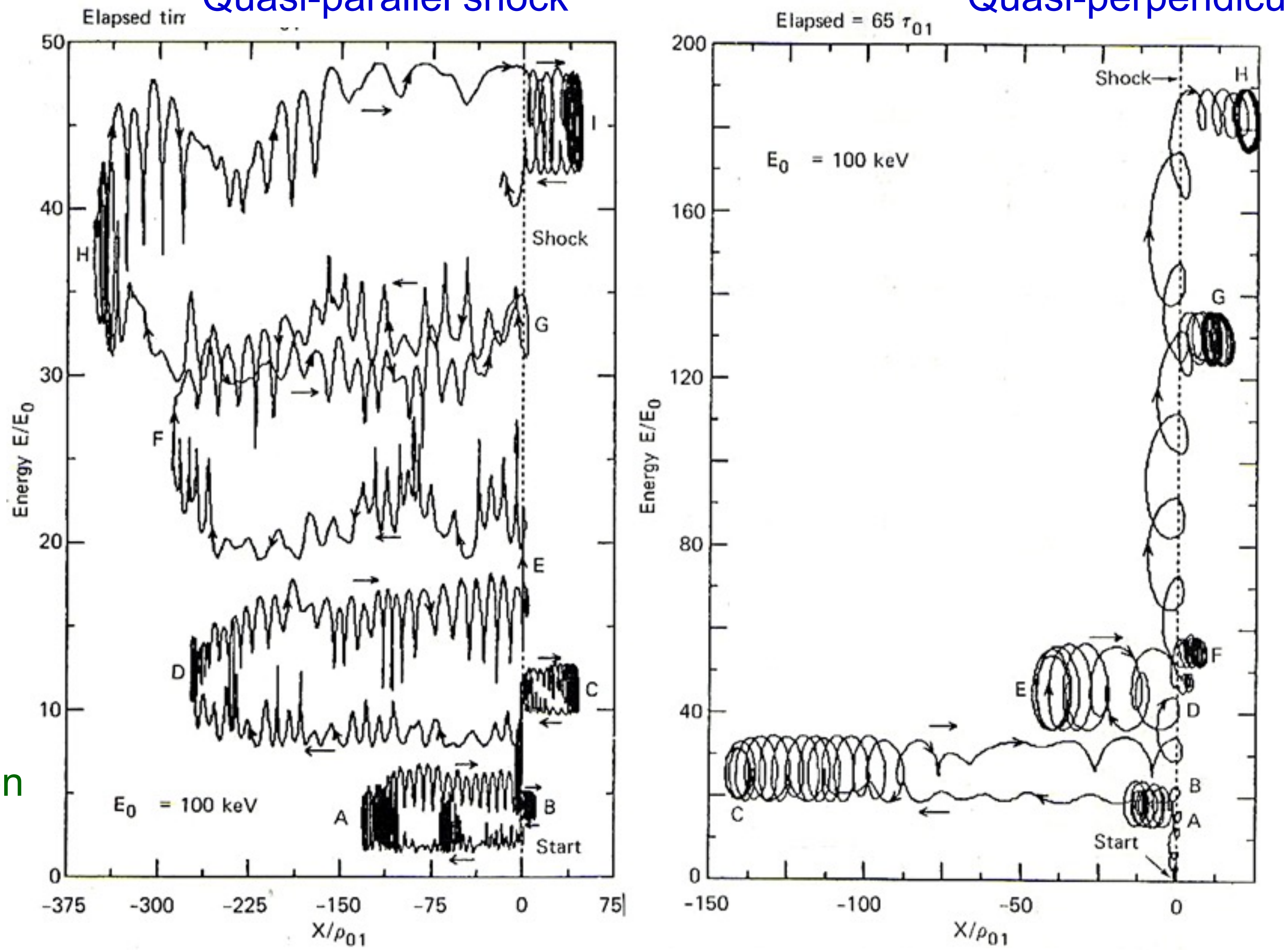


Simulated arrival locations of particles as a function of longitude and latitude and time

The basic physics of particle acceleration at shocks

Quasi-parallel shock

Quasi-perpendicular shock



Increasing Particle Energy

Slower Acceleration case

More rapid acceleration

Position relative to the shock

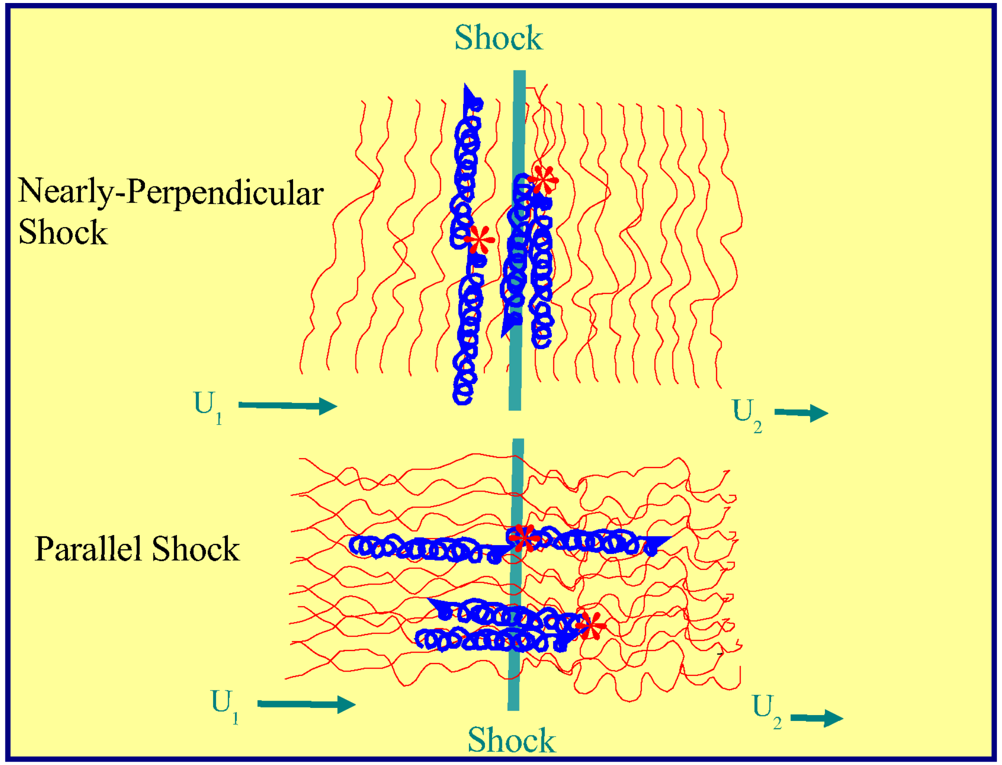
Decker, 1988

Acceleration at Shocks:
Diffusive Shock Acceleration Theory 101

Axford et al. (1977), Krymsky, (1977), Bell (1978), Blandford & Ostriker (1978)

- Charged particles that are confined near the strong plasma compression associated with a collisionless shock have a net gain in energy

- The confinement is due to scattering within fluctuating magnetic fields, due either to pre-existing turbulence through which the shock moves, or those due to instabilities associated with the shock



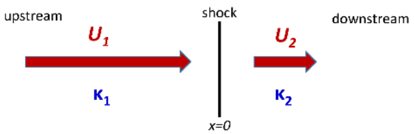
Quantitative solution for the distribution function of accelerated particles comes from the Parker equation

$$\frac{\partial f}{\partial t} = -V_{w,i}\frac{\partial f}{\partial x_i} + \frac{\partial}{\partial x_i}\kappa_{ij}\frac{\partial f}{\partial x_j} - V_{D,i}\frac{\partial f}{\partial x_i} + \frac{1}{3}\frac{\partial V_{w,i}}{\partial x_i}\frac{\partial f}{\partial \ln p} + Q$$

advection ***diffusion*** ***drift*** ***energy change***

Under the Hood: The solution of the Parker equation for a shock

Consider a shock



We wish to solve the Parker equation

$$\frac{\partial f(x,p)}{\partial t} = \frac{\partial}{\partial x}\left[\kappa(p)\frac{\partial f(x,p)}{\partial x}\right] - U(x)\frac{\partial f(x,p)}{\partial x} + \frac{1}{3}\frac{dU(x)}{dx}p\frac{\partial f(x,p)}{\partial p} = 0$$

Solve this separately in the upstream (1) and downstream (2) regions.

$$\frac{\partial}{\partial x}\left[\kappa_{1,2}(p)\frac{\partial f_{1,2}(x,p)}{\partial x}\right] - U_{1,2}\frac{\partial f_{1,2}(x,p)}{\partial x} = 0$$

This has the general solution

$$f_{1,2}(x,p) = A_{1,2}(p)\exp\left[{}^{U_{1,2}x}/_{\kappa_{1,2}}\right] + B_{1,2}(p)$$

Subject to the following boundary conditions / constraints

1. $f_1(-\infty,p) = f_S(p) \Longrightarrow B_1(p) = f_S(p)$
2. $f_2(+\infty,p)$ *is finite* $\Longrightarrow A_2(p) = 0$
3. $f_1(0,p) = f_2(0,p) \Longrightarrow A_1(p) + f_S(p) = B_2(p)$

Giving:

$$\boxed{\begin{array}{ll} f_1(x,p) = A_1(p)\exp\left[{}^{U_1x}/_{\kappa_1}\right] + f_S(p) & x<0 \\ f_2(x,p) = A_1(p) + f_S(p) & x\geq 0 \end{array}} \qquad (1)$$

To get $A_1(p)$, we integrate over small region near shock, giving

$$\lim_{\epsilon\to 0}\int_{-\epsilon}^{+\epsilon}\left[\underbrace{\frac{\partial}{\partial x}\left[\kappa(p)\frac{\partial f(x,p)}{\partial x}\right]}_{\text{Term 1}} - \underbrace{U(x)\frac{\partial f(x,p)}{\partial x}}_{\text{Term 2}} + \underbrace{\frac{1}{3}\frac{dU(x)}{dx}p\frac{\partial f(x,p)}{\partial p}}_{\text{Term 3}}\right]dx = 0$$

Term 1

$$\lim_{\epsilon\to 0}\left[\cancelto{0}{\kappa_2(p)\frac{\partial f_2(x,p)}{\partial x}}\right]_{x=+\epsilon} - \lim_{\epsilon\to 0}\left[\kappa_1(p)\frac{\partial f_1(x,p)}{\partial x}\right]_{x=-\epsilon} = -\lim_{\epsilon\to 0}\left[A_1(p)U_1\exp\left({}^{U_1x}/_{\kappa_1}\right)\right]_{x=-\epsilon}$$

$$= -A_1(p)U_1$$

Term 2

$$\lim_{\epsilon\to 0}\Big[U(x)f(x,p)\Big]_{x=-\epsilon}^{x=+\epsilon} = 0$$

Term 3

$$\lim_{\epsilon\to 0}\int_{-\epsilon}^{+\epsilon}\left[\frac{1}{3}(U_2-U_1)\delta(x)p\frac{\partial f(x,p)}{\partial p}\right]dx = \frac{1}{3}(U_2-U_1)p\frac{\partial f_2(p)}{\partial p} = \frac{U_2-U_1}{3}p\frac{\partial}{\partial p}(A_1(p)+f_S(p))$$

Combining the three terms leads to

$$-A_1(p)U_1 + \frac{U_2-U_1}{3}\frac{dA_1(p)}{d\ln p} + \frac{U_2-U_1}{3}\frac{df_S(p)}{d\ln p} = 0$$

Re-arranging, leads to

$$\frac{dA_1(p)}{d\ln p} + \alpha A_1(p) = -\frac{df_S(p)}{d\ln p} \qquad \text{where: } \alpha = \frac{3U_1}{U_1-U_2}$$

Defining a new variable, $y = \ln p$, it can be shown

$$\exp(-\alpha y)\frac{d}{dy}[\exp(\alpha y)A_1(y)] = -\frac{df_S(y)}{dy}$$

$$\Longrightarrow \frac{d}{dy}[\exp(\alpha y)A_1(y)] = -\exp(\alpha y)\frac{df_S(y)}{dy}$$

$$\Longrightarrow \exp(\alpha y)A_1(y) + C = -\int\exp(\alpha y)\frac{df_S(y)}{dy}dy$$

$$\Longrightarrow A_1(y) = -\exp(-\alpha y)\int\exp(\alpha y)\frac{df_S(y)}{dy}dy - C\exp(-\alpha y)$$

In terms of p, this is

$$A_1(p) = -p^{-\alpha}\int p^{\alpha}\frac{df_S(p)}{dp}dp - Cp^{-\alpha}$$

Since A_1 must remain finite as $p\to 0$, we require $C = 0$.
Integrating by parts, and taking $f_S\to 0$ as $p\to 0$, it can be shown

$$A_1(p) = -p^{-\alpha}\left[p^{\alpha}f_S(p) - \alpha\int p^{\alpha-1}f_S(p)dp\right] = -f_S(p) + \alpha p^{-\alpha}\int$$

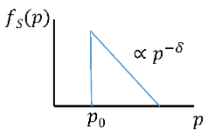
Inserting this into (1) on slide 6, we have, finally,

$$\boxed{\begin{array}{ll} f_2(p) = \alpha p^{-\alpha}\int p^{\alpha-1}f_S(p)dp & x\geq 0 \\ f_1(p) = f_S(p) + [f_2(p) - f_S(p)]\mathrm{e} & \end{array}}$$

Case 1: $f_S(p) = Q_0\delta(p-p_0)$

$$f_2(p) = \alpha Q_0 p^{-\alpha}p_0^{\alpha-1} = \frac{\alpha Q_0}{p_0}\left(\frac{p}{p_0}\right)^{-\alpha} \longleftarrow \text{The standard DSA result}$$

Case 2: $f_S(p) = \begin{cases} 0 & p<0 \\ f_S(p_0)\left(\frac{p}{p_0}\right)^{-\delta} & p\geq p_0 \end{cases}$



$$f_2(p) = \frac{\alpha}{\alpha-\delta}f_S(p_0)\left[\left(\frac{p}{p_0}\right)^{-\delta} - \left(\frac{p}{p_0}\right)^{-\alpha}\right] \qquad \alpha\neq\delta$$

$$f_2(p) = \alpha f_S(p_0)\left(\frac{p}{p_0}\right)^{-\alpha}\ln\left(\frac{p}{p_0}\right) \qquad \alpha=\delta$$

These slides are included in the list of "extras"

Solution to the Parker equation for a planar shock and mono-energetic source at the shock

$$f(x,p) = \begin{cases} f_0\left(\frac{p}{p_0}\right)^{-\gamma} \exp\left(-\frac{U_1|x|}{\kappa_{xx,1}(p)}\right) & x < 0 \\ f_0\left(\frac{p}{p_0}\right)^{-\gamma} & x \geq 0 \end{cases}$$

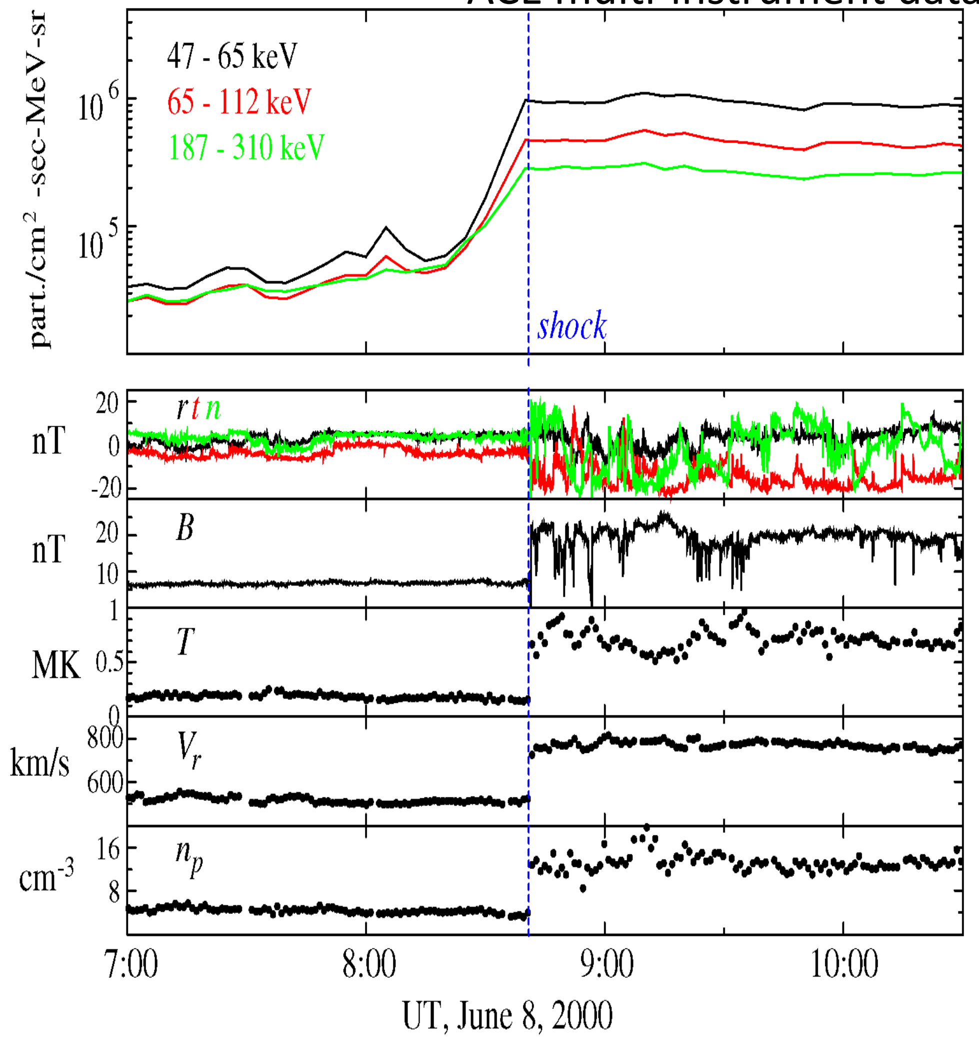
where $\gamma = 3U_1/(U_1 - U_2) = 3r/(r-1)$

The diffusion coefficient in this illustrative example is normal to the shock, which is related to the magnetic field

$$\kappa_{xx} = \kappa_\perp \sin^2\theta_{Bn} + \kappa_\parallel \cos^2\theta_{Bn}$$

Qualitatively consistent with spacecraft observations

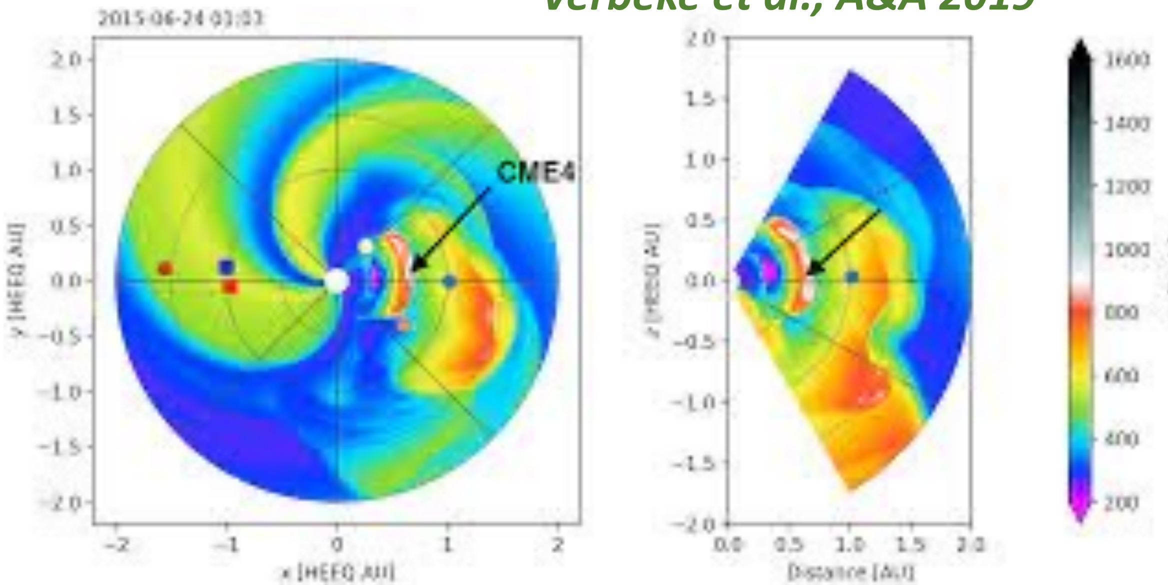
ACE multi-instrument data

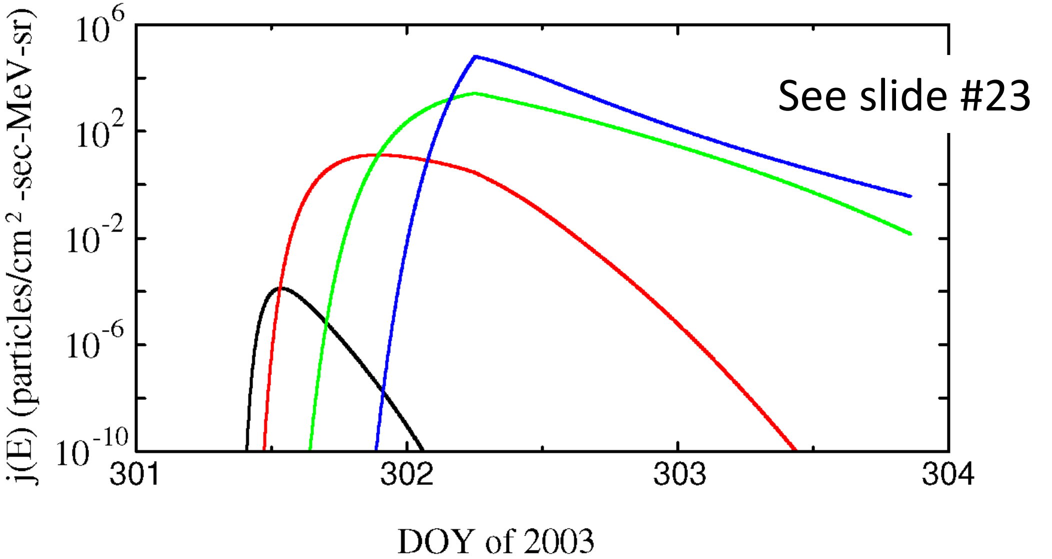


A comment about combining large-scale MHD models of CMEs with particle –acceleration/transport models

- To properly solve particle acceleration at a shock, the diffusive length scale MUST be resolved by the MHD model.
  - MHD grid cell size < K_{xx}/U_1
  - This is a very stringent constraint
- *in situ* observations of SEPs coincident with CME-shocks at 1AU finds this length scale to be about 0.002AU for 100 keV protons.
- It is MUCH shorter near the Sun because the magnetic field is much stronger there. Perhaps 100-1000 times shorter.
- Thus, MHD models (that are combined with SEP models) must resolve length scales as small as 10^{-5} AU or about 0.01 solar radii.

Verbeke et al., A&A 2019



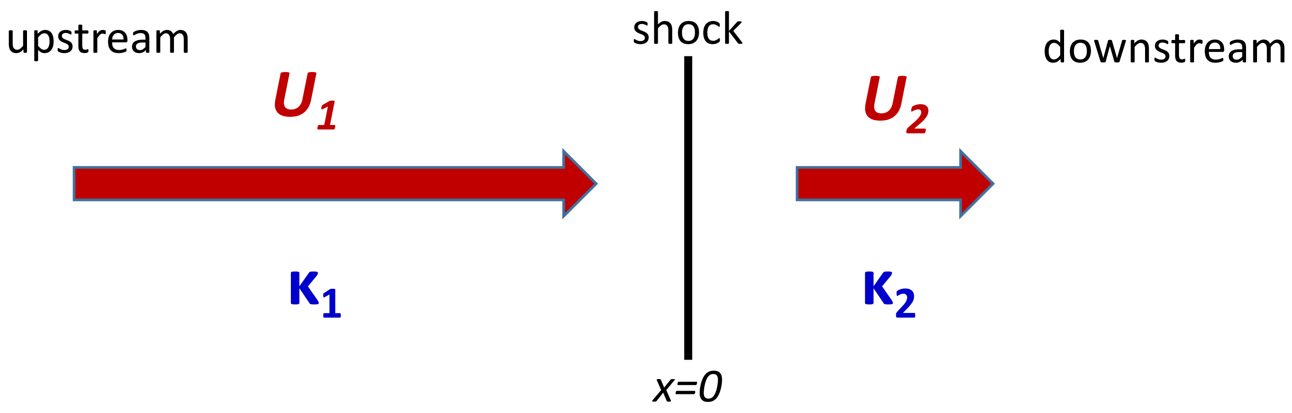


Final comments

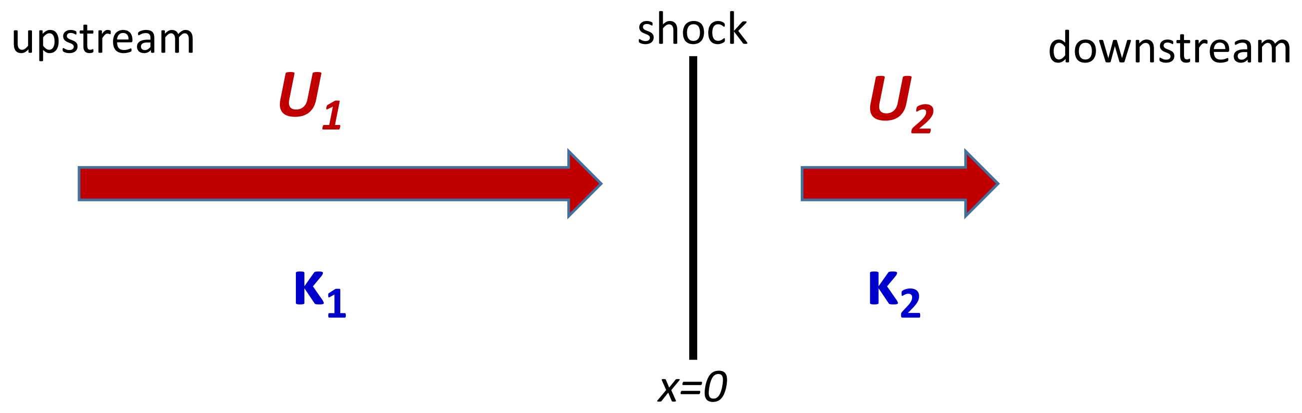
- SEPs are a very important aspect of the space radiation environment, and it is important to understand how they are accelerated and transport in space.
- This underlying physics of particle transport and acceleration is reasonably well established, but the equations are not “trivial” to solve.
- It is very important to recognize the underlying assumptions in any model, and this is particularly important in SEP acceleration/transport models as well.
- Combining large-scale models of CMEs with SEP models, in a manner that includes most of the important physics, is a <u>***GRAND CHALLENGE***</u>

extras

Consider a shock



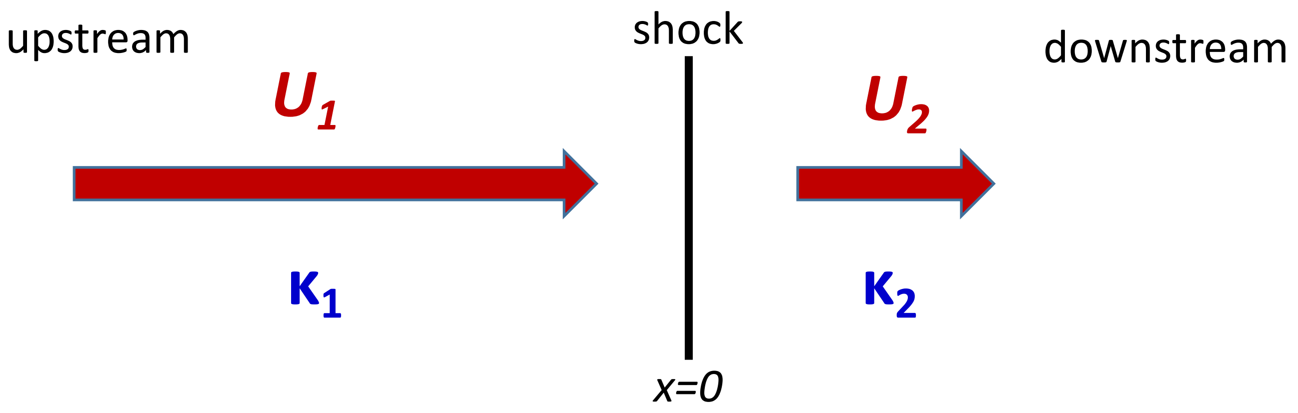
Consider a shock



We wish to solve the Parker equation

$$\frac{\partial f(x,p)}{\partial t} = \frac{\partial}{\partial x}\left[\kappa(p)\frac{\partial f(x,p)}{\partial x}\right] - U(x)\frac{\partial f(x,p)}{\partial x} + \frac{1}{3}\frac{dU(x)}{dx}p\frac{\partial f(x,p)}{\partial p} = 0$$

Consider a shock



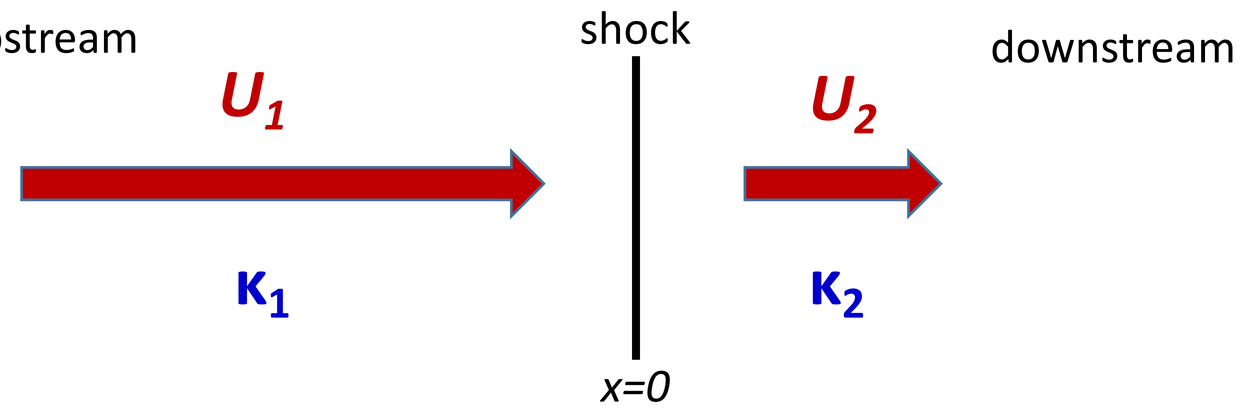
We wish to solve the Parker equation

$$\frac{\partial f(x,p)}{\partial t} = \frac{\partial}{\partial x}\left[\kappa(p)\frac{\partial f(x,p)}{\partial x}\right] - U(x)\frac{\partial f(x,p)}{\partial x} + \frac{1}{3}\frac{dU(x)}{dx}p\frac{\partial f(x,p)}{\partial p} = 0$$

Solve this separately in the upstream (1) and downstream (2) regions.

$$\frac{\partial}{\partial x}\left[\kappa_{1,2}(p)\frac{\partial f_{1,2}(x,p)}{\partial x}\right] - U_{1,2}\frac{\partial f_{1,2}(x,p)}{\partial x} = 0$$

Consider a shock



We wish to solve the Parker equation

$$\frac{\partial f(x,p)}{\partial t} = \frac{\partial}{\partial x}\left[\kappa(p)\frac{\partial f(x,p)}{\partial x}\right] - U(x)\frac{\partial f(x,p)}{\partial x} + \frac{1}{3}\frac{dU(x)}{dx}p\frac{\partial f(x,p)}{\partial p} = 0$$

Solve this separately in the upstream (1) and downstream (2) regions.

$$\frac{\partial}{\partial x}\left[\kappa_{1,2}(p)\frac{\partial f_{1,2}(x,p)}{\partial x}\right] - U_{1,2}\frac{\partial f_{1,2}(x,p)}{\partial x} = 0$$

This has the general solution

$$f_{1,2}(x,p) = A_{1,2}(p)\exp\left[{U_{1,2}\,x}/{\kappa_{1,2}}\right] + B_{1,2}(p)$$

Subject to the following boundary conditions / constraints

1. $f_1(-\infty, p) = f_S(p) \Longrightarrow B_1(p) = f_S(p)$
2. $f_2(+\infty, p)$ *is finite* $\Longrightarrow A_2(p) = 0$
3. $f_1(0, p) = f_2(0, p) \Longrightarrow A_1(p) + f_S(p) = B_2(p)$

Giving:

$$\boxed{\begin{array}{ll} f_1(x,p) = A_1(p)\exp\left[{U_1 x}/{\kappa_1}\right] + f_S(p) & x < 0 \\ f_2(x,p) = A_1(p) + f_S(p) & x \geq 0 \end{array}} \qquad (1)$$

To get $A_1(p)$, we integrate over small region near shock, giving

$$\lim_{\epsilon \to 0} \int_{-\epsilon}^{+\epsilon} \left[\underbrace{\frac{\partial}{\partial x}\left[\kappa(p)\frac{\partial f(x,p)}{\partial x}\right]}_{\textbf{Term 1}} - \underbrace{U(x)\frac{\partial f(x,p)}{\partial x}}_{\textbf{Term 2}} + \underbrace{\frac{1}{3}\frac{dU(x)}{dx}p\frac{\partial f(x,p)}{\partial p}}_{\textbf{Term 3}} \right] dx = 0$$

Term 1

$$\lim_{\epsilon\to 0}\left[\kappa_2(p)\frac{\partial f_2(x,p)}{\partial x}\right]_{x=+\epsilon}^{\;\to 0} - \lim_{\epsilon\to 0}\left[\kappa_1(p)\frac{\partial f_1(x,p)}{\partial x}\right]_{x=-\epsilon} = -\lim_{\epsilon\to 0}\left[A_1(p)U_1\exp\left(U_1 x/\kappa_1\right)\right]_{x=-\epsilon}$$

$$= -A_1(p)U_1$$

Term 2

$$\lim_{\epsilon\to 0}\Big[U(x)f(x,p)\Big]_{x=-\epsilon}^{x=+\epsilon} = 0$$

Term 3

$$\lim_{\epsilon\to 0}\int_{-\epsilon}^{+\epsilon}\left[\frac{1}{3}(U_2-U_1)\delta(x)p\frac{\partial f(x,p)}{\partial p}\right]dx = \frac{1}{3}(U_2-U_1)p\frac{\partial f_2(p)}{\partial p} = \frac{U_2-U_1}{3}p\frac{\partial}{\partial p}\left(A_1(p)+f_S(p)\right)$$

Combining the three terms leads to

$$-A_1(p)U_1 + \frac{U_2 - U_1}{3}\frac{dA_1(p)}{d\ln p} + \frac{U_2 - U_1}{3}\frac{df_S(p)}{d\ln p} = 0$$

Re-arranging, leads to

$$\frac{dA_1(p)}{d\ln p} + \alpha A_1(p) = -\frac{df_S(p)}{d\ln p}$$

where: $\alpha = \frac{3U_1}{U_1 - U_2}$

Defining a new variable, $y = \ln p$, it can be shown

$$\exp(-\alpha y)\frac{d}{dy}[\exp(\alpha y)\,A_1(y)] = -\frac{df_S(y)}{dy}$$

$$\Longrightarrow \quad \frac{d}{dy}[\exp(\alpha y)\,A_1(y)] = -\exp(\alpha y)\frac{df_S(y)}{dy}$$

$$\Longrightarrow \quad \exp(\alpha y)\, A_1(y) + C = -\int \exp(\alpha y) \frac{df_S(y)}{dy} dy$$

$$\Longrightarrow \quad A_1(y) = -\exp(-\alpha y) \int \exp(\alpha y) \frac{df_S(y)}{dy} dy - C \exp(-\alpha y)$$

In terms of *p*, this is

$$A_1(p) = -p^{-\alpha} \int p^{\alpha} \frac{df_S(p)}{dp} dp - C p^{-\alpha}$$

Since A_1 must remain finite as $p \rightarrow 0$, we require $C = 0$.
Integrating by parts, and taking $f_S \rightarrow 0$ as $p \rightarrow 0$, it can be shown

$$A_1(p) = -p^{-\alpha} \left[p^{\alpha} f_S(p) - \alpha \int p^{\alpha-1} f_S(p) dp \right] = -fS(p) + \alpha\, p^{-\alpha} \int p^{\alpha-1} f_S(p) dp$$

Inserting this into (1) on slide 6, we have, finally,

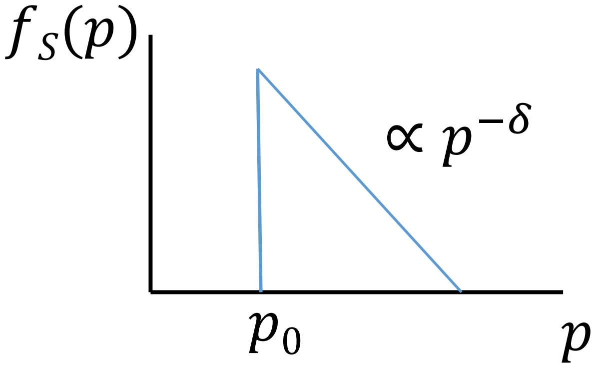
$$\boxed{\begin{array}{ll} f_2(p) = \alpha\, p^{-\alpha} \int p^{\alpha-1} f_S(p)\,dp & x \geq 0 \\ f_1(p) = f_S(p) + [f_2(p) - f_S(p)]\exp\left({}^{U_1\,x}/_{\kappa_1}\right) & x < 0 \end{array}}$$

Case 1: $f_S(p) = Q_0 \delta(p - p_0)$

$$f_2(p) = \alpha Q_0 p^{-\alpha} {p_0}^{\alpha-1} = \frac{\alpha Q_0}{p_0}\left(\frac{p}{p_0}\right)^{-\alpha}$$

The standard DSA result

Case 2: $$f_S(p) = \begin{cases} 0 & p < 0 \\ f_S(p_0)\left(\frac{p}{p_0}\right)^{-\delta} & p \geq p_0 \end{cases}$$



$$f_2(p) = \frac{\alpha}{\alpha - \delta} f_S(p_0)\left[\left(\frac{p}{p_0}\right)^{-\delta} - \left(\frac{p}{p_0}\right)^{-\alpha}\right] \qquad \alpha \neq \delta$$

$$f_2(p) = \alpha f_S(p_0)\left(\frac{p}{p_0}\right)^{-\alpha} \ln\left(\frac{p}{p_0}\right) \qquad \alpha = \delta$$

The solution is (see also *Neergaard-Parker & Zank*, 2012):

$$f_2(p) = \alpha\, p^{-\alpha} \int p^{\alpha-1} f_S(p)\,dp \qquad x \geq 0$$

$$f_1(p) = f_S(p) + [f_2(p) - f_S(p)]\exp\left({}^{U_1 x}/_{\kappa_1}\right) \qquad x < 0$$

where $\alpha = \frac{3U_1}{U_1 - U_2}$

For the specific case in which the source is a power-law distribution, with spectral index, δ, we obtain for the *downstream distribution*:

$$\boxed{f_2(p) = \frac{\alpha}{\alpha-\delta} f_S(p_0)\left[\left(\frac{p}{p_0}\right)^{-\delta} - \left(\frac{p}{p_0}\right)^{-\alpha}\right]}$$

$U_1/U_2 = 4$

$\alpha = 4$