# Aspects of the Physics of High-Energy Charged Particles in the Heliosphere

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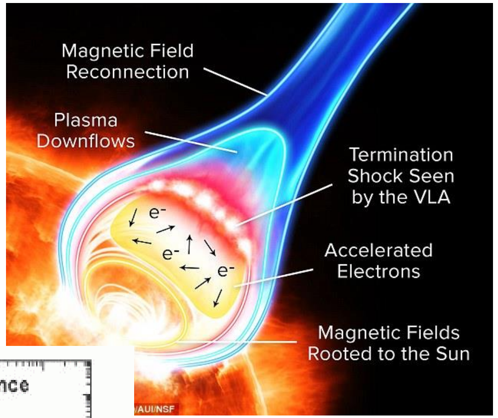
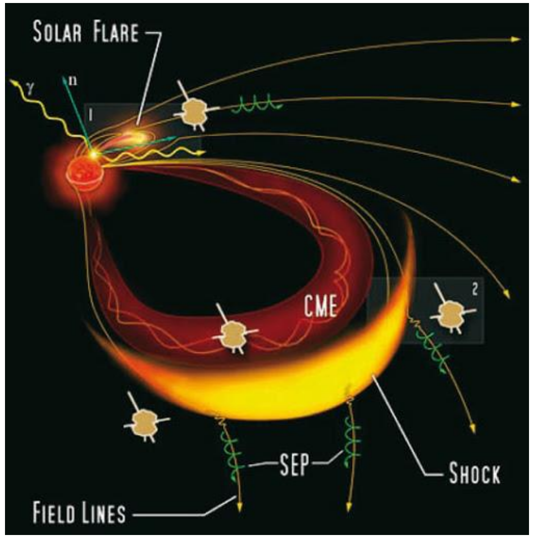
*Heliophysics Summer School, June 22, 2021*

# Motivation

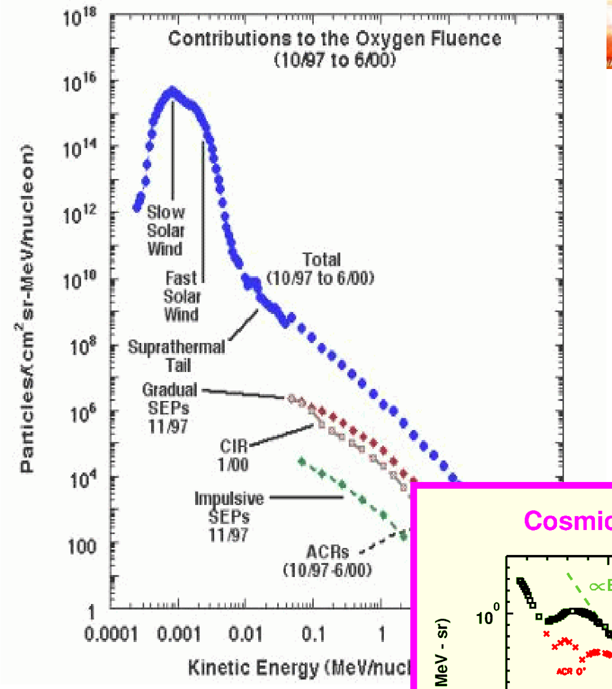
- The origin of the high-energy particles and cosmic rays is an important and unsolved problems in astrophysics and heliophysics.
- The Sun and heliosphere are significant sources of energetic particles. Excellent “laboratories” for studying the physics of particle acceleration and transport, particularly at shocks *in situ*.
- Many unsolved problems remain, and there are numerous observations to analyze.

Flare termination shock

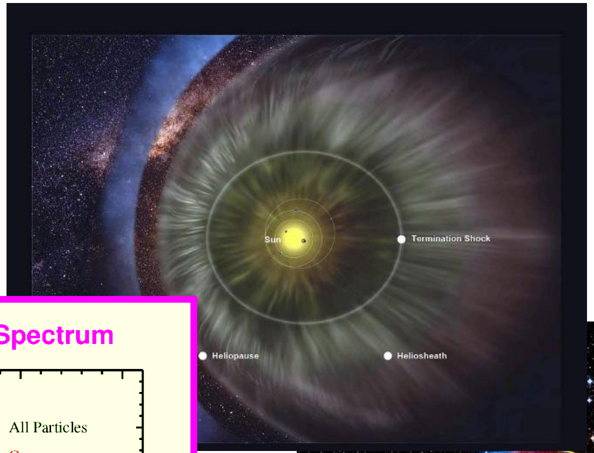


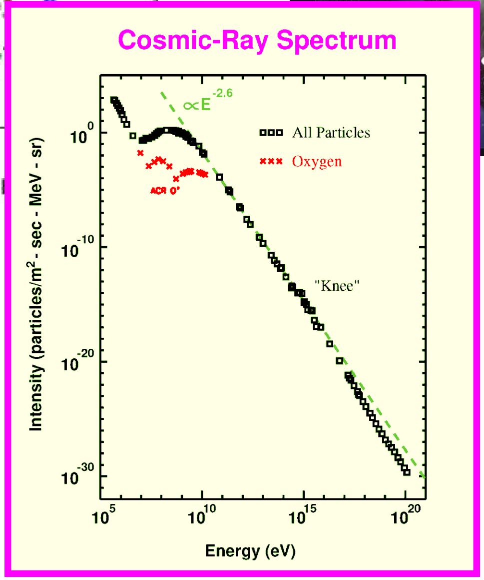




CME shock

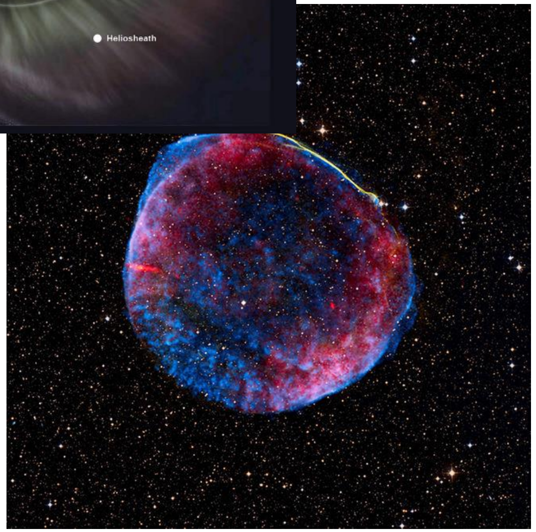



Termination Shock



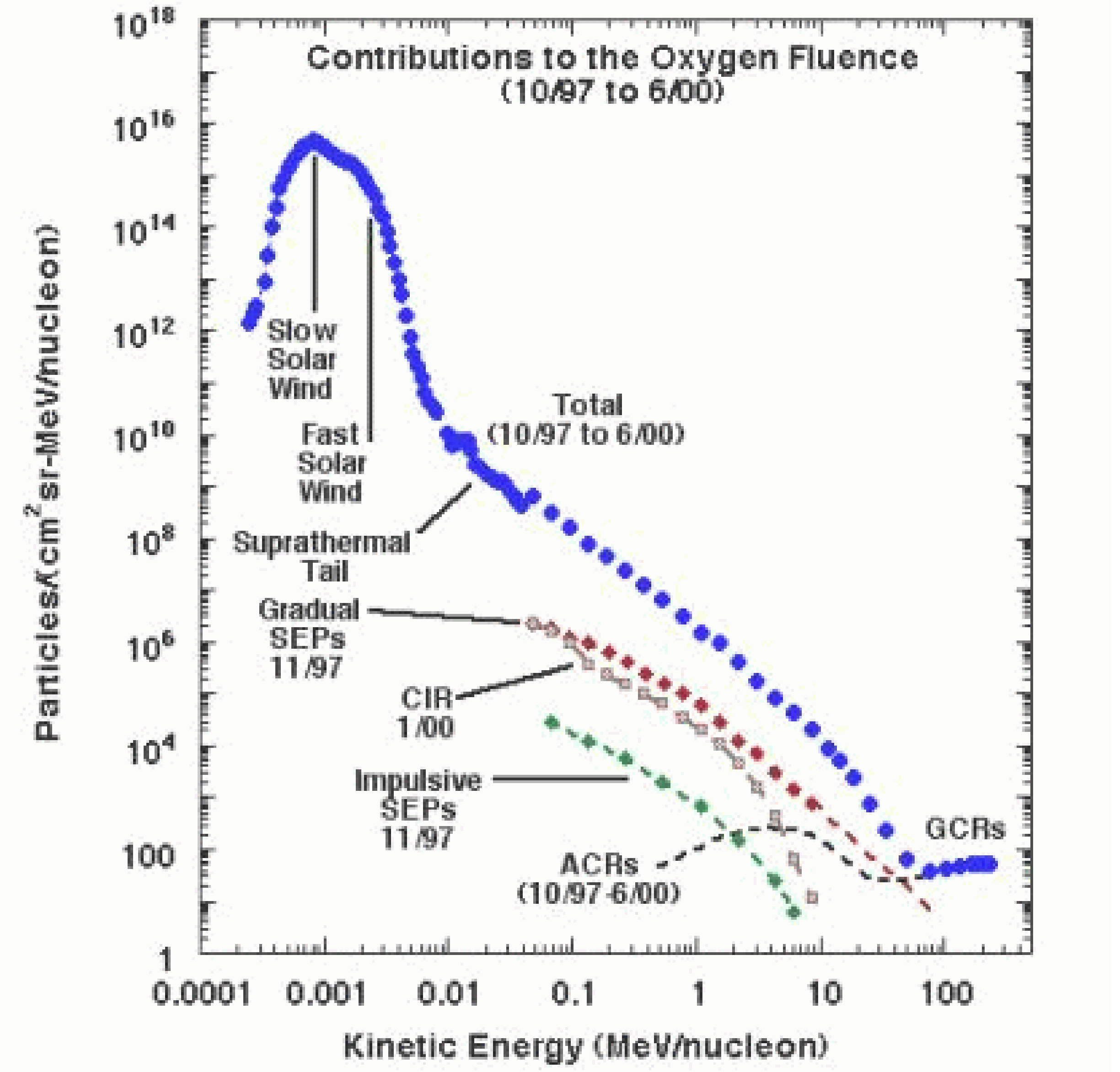

SNR blast wave



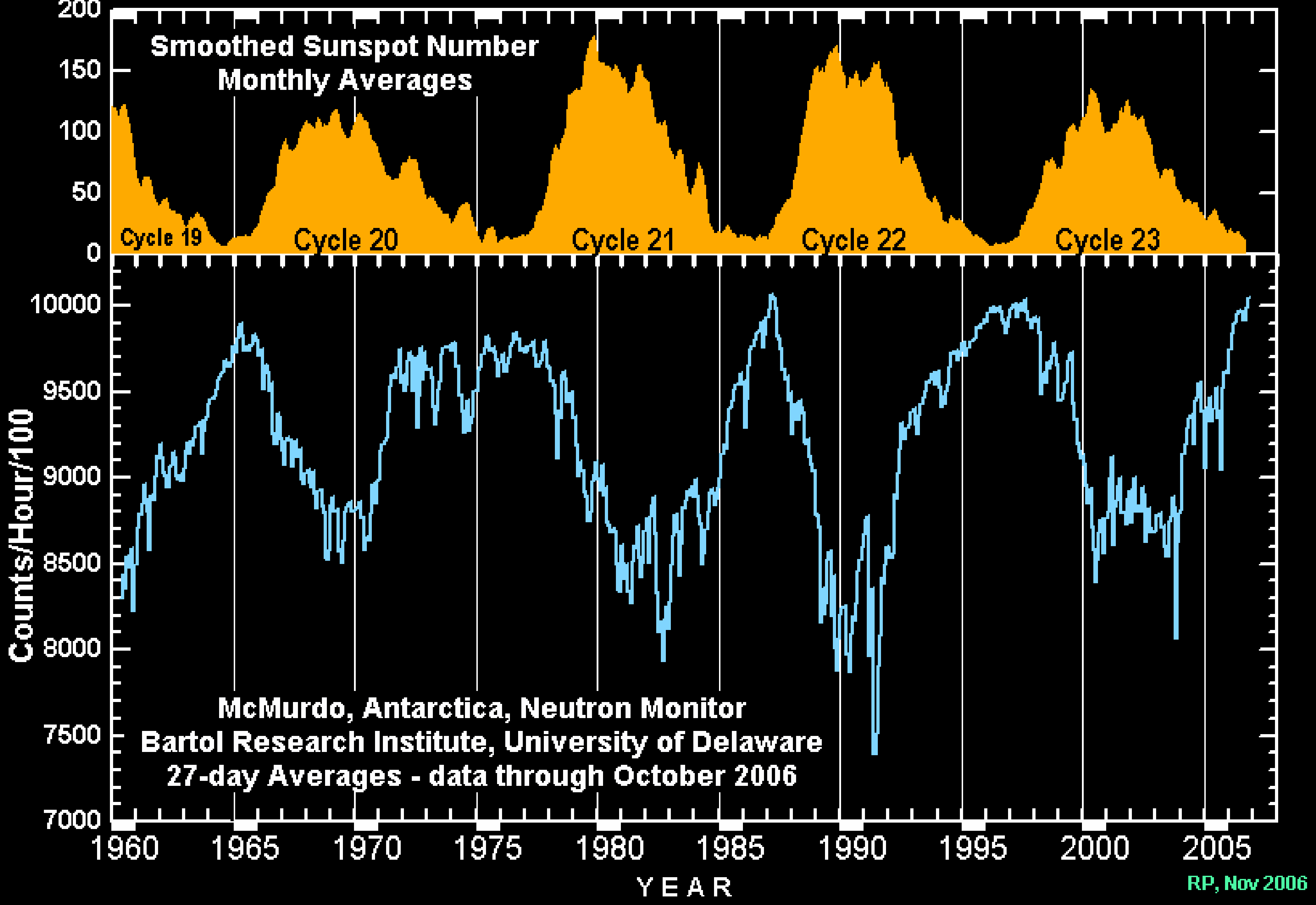
# High-energy charged particles in the solar system

- Galactic cosmic rays(GCRs)
- Anomalous cosmic rays (ACRs)
- Solar energetic particles (SEPs)
  - High-intensity, long duration events associated with coronal mass ejections (Gradual SEPs)
  - Lower intensity, short lived events associated with brief, impulsive solar flares (Impulsive SEPs)
  - Recurrent events associated with solar wind structures that co-rotate with the Sun (CIRs)
  - Solar Proton Events (SPEs) are a subclass of SEPs associated the highest intensity events that are defined by the NOAA/GOES.
  - Ground-level enhancements are extremely intense SEP events.
- High-energy ions and electrons from planetary magnetospheres
  - Jovian electrons
  - Aurora and magnetospheric "substorms"

*Data from NASAs Advanced Composition Explorer spacecraft*

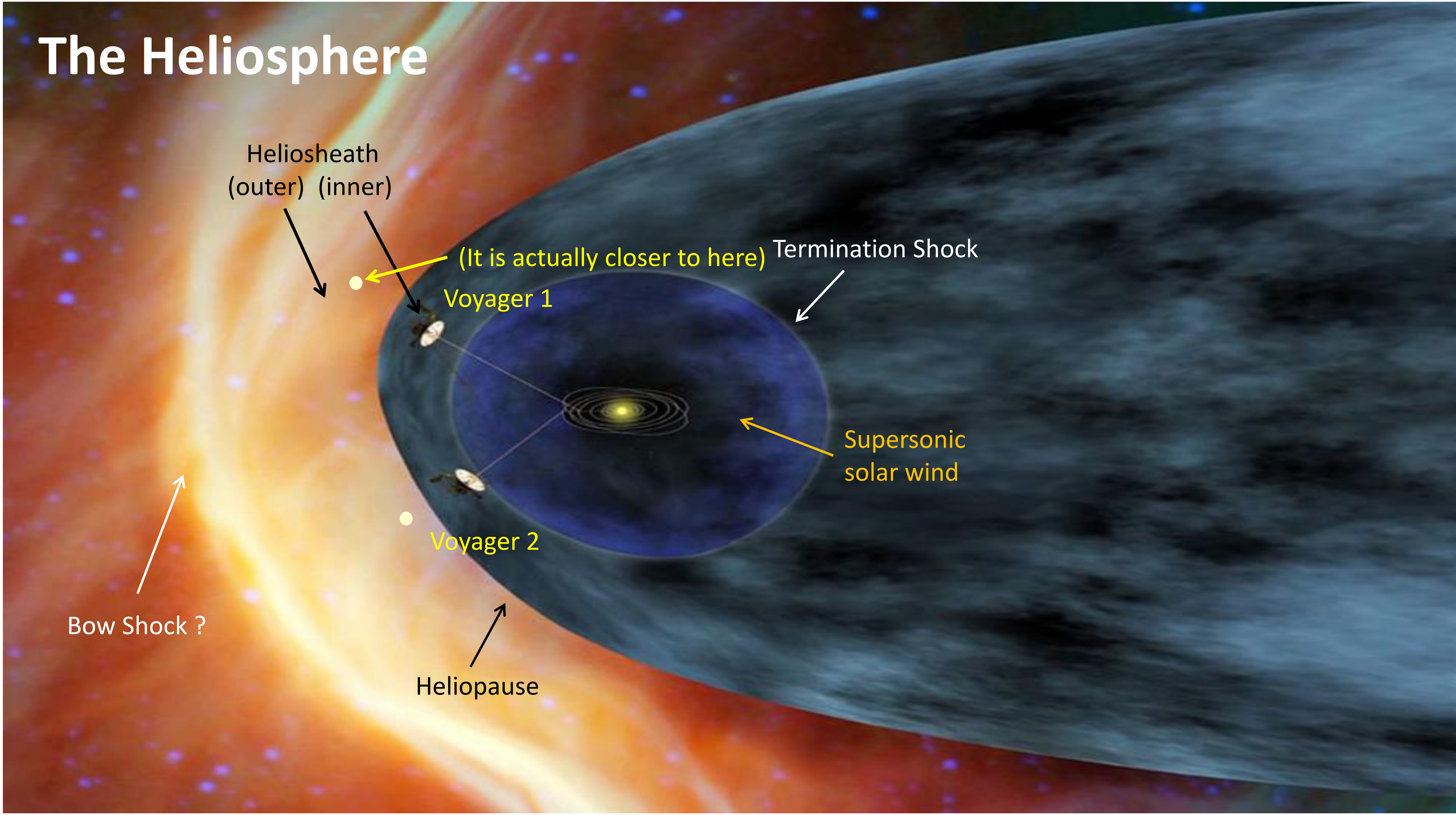


# Galactic Cosmic Rays and the Solar Cycle

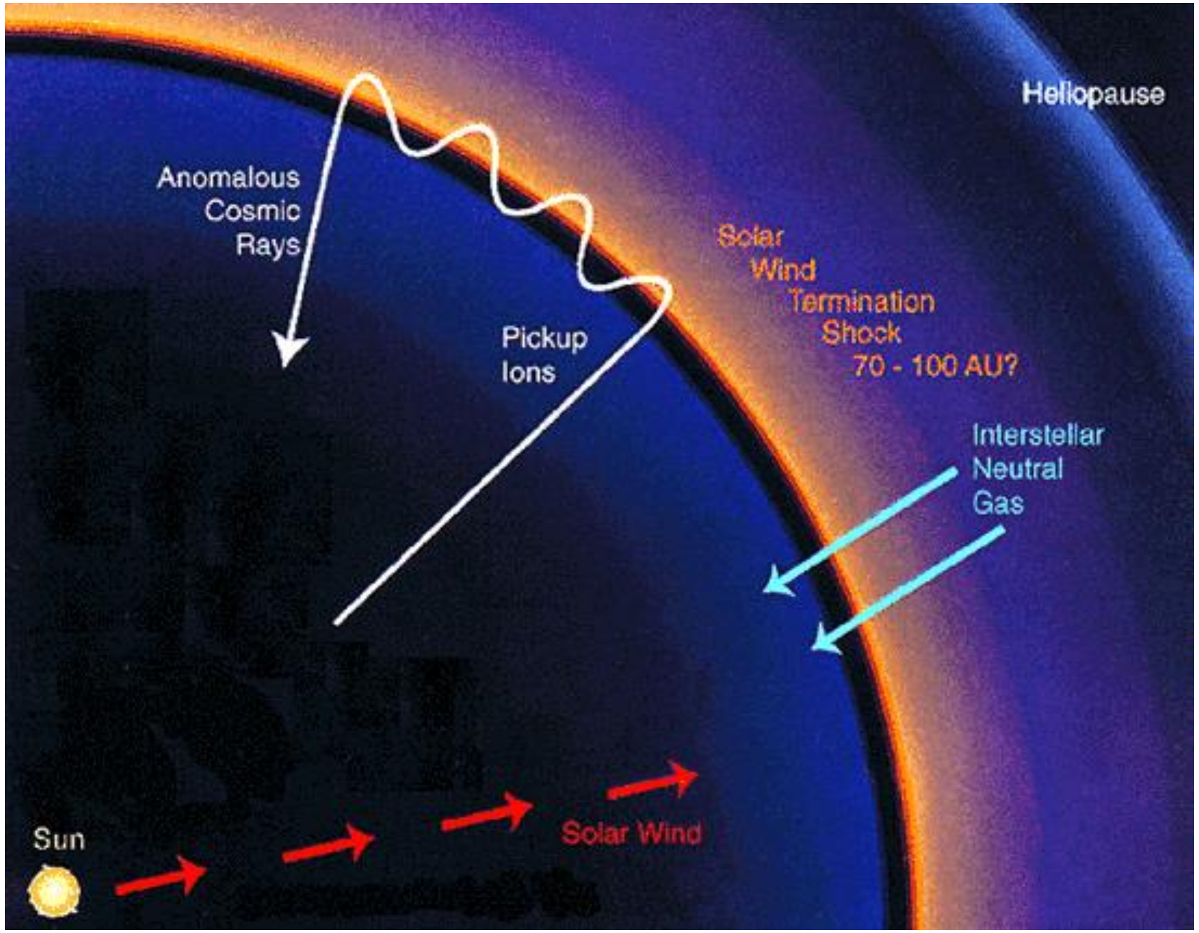


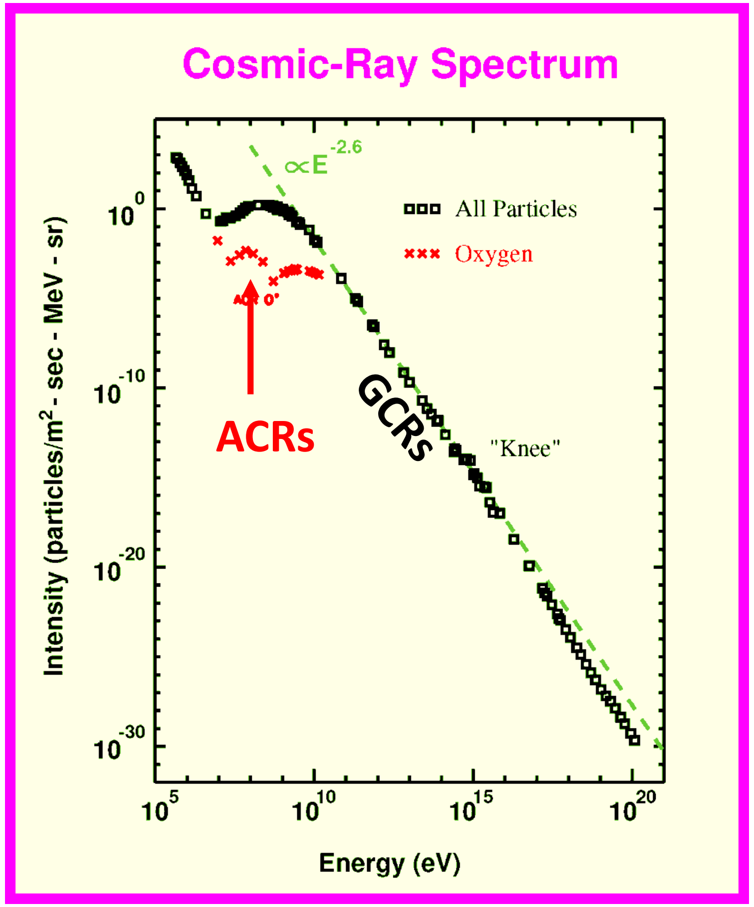
# Cosmic Rays on Earth Since 1964

# The Heliosphere

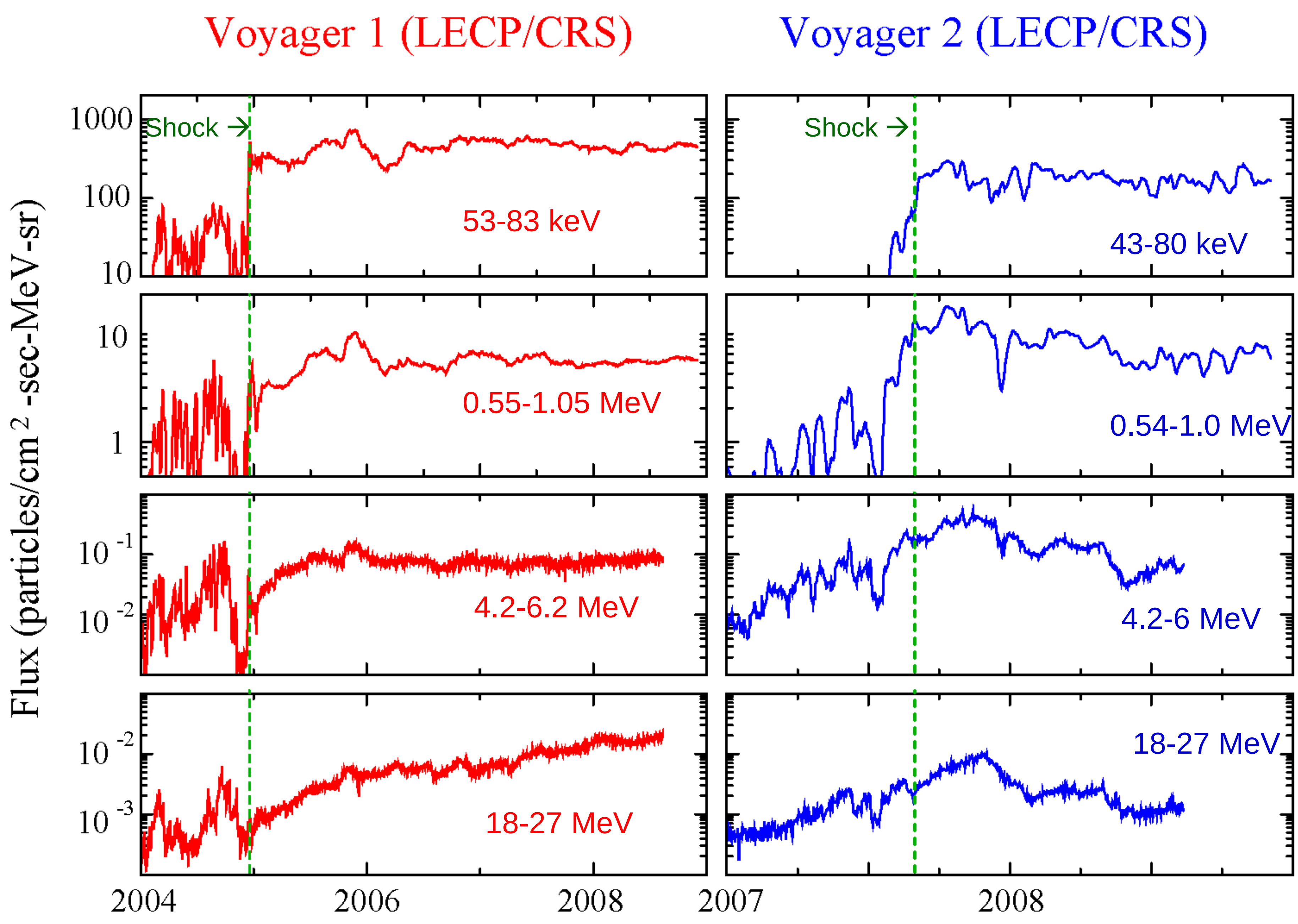


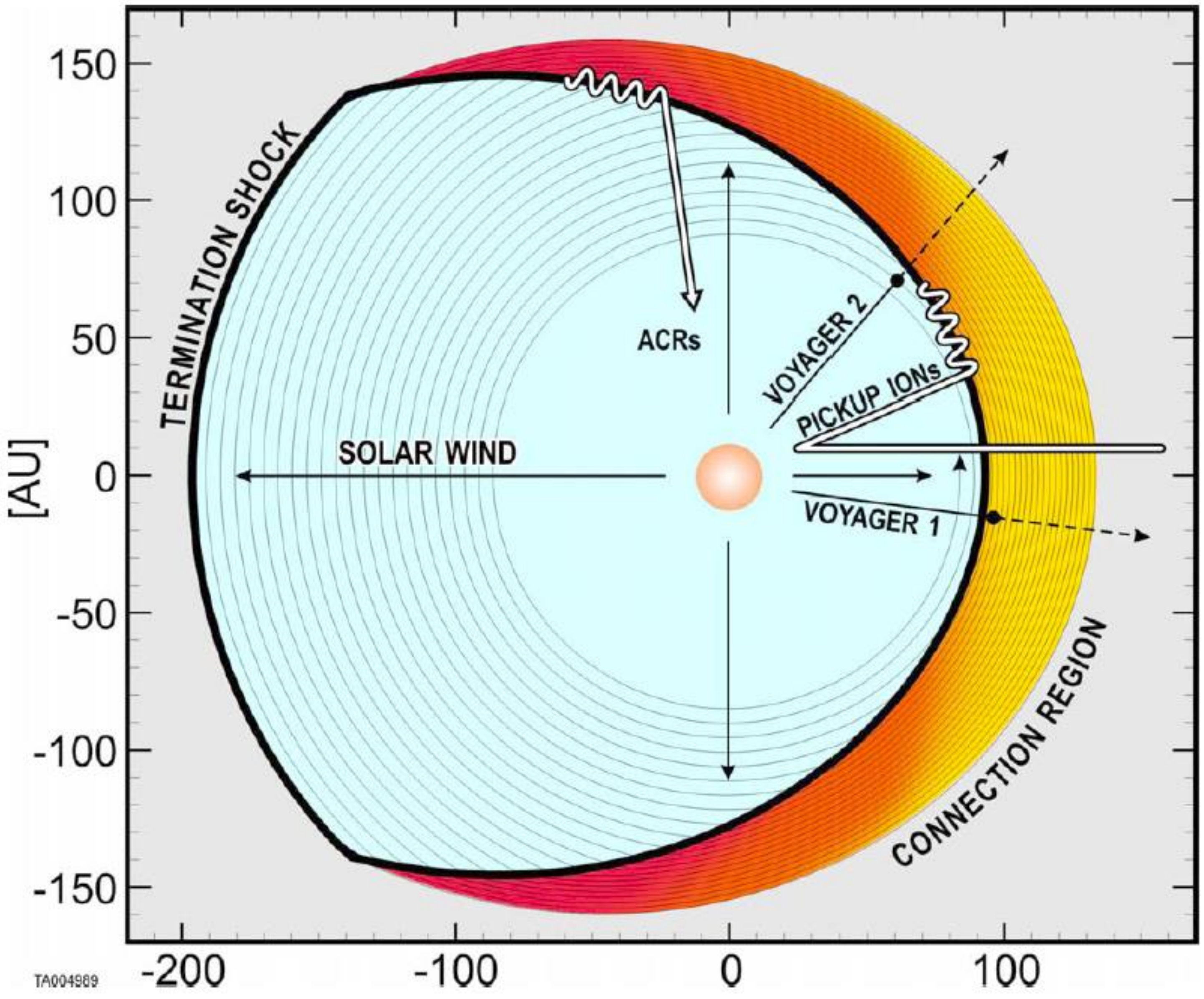
# Particle Acceleration at the Heliosphere's termination shock




***Anomalous Cosmic Rays (ACRs) are accelerated interstellar pickup ions, most likely the result of diffusive shock acceleration at the solar-wind termination shock***


Voyager 1 (LECP/CRS)
Voyager 2 (LECP/CRS)
Shock →
Shock →
Flux (particles/cm $^2$ -sec-MeV-sr)
1000
100
10
1
$10^{-1}$
$10^{-2}$
$10^{-3}$
53-83 keV
0.55-1.05 MeV
4.2-6.2 MeV
18-27 MeV
43-80 keV
0.54-1.0 MeV
4.2-6 MeV
18-27 MeV
2004
2006
2008
2007
2008

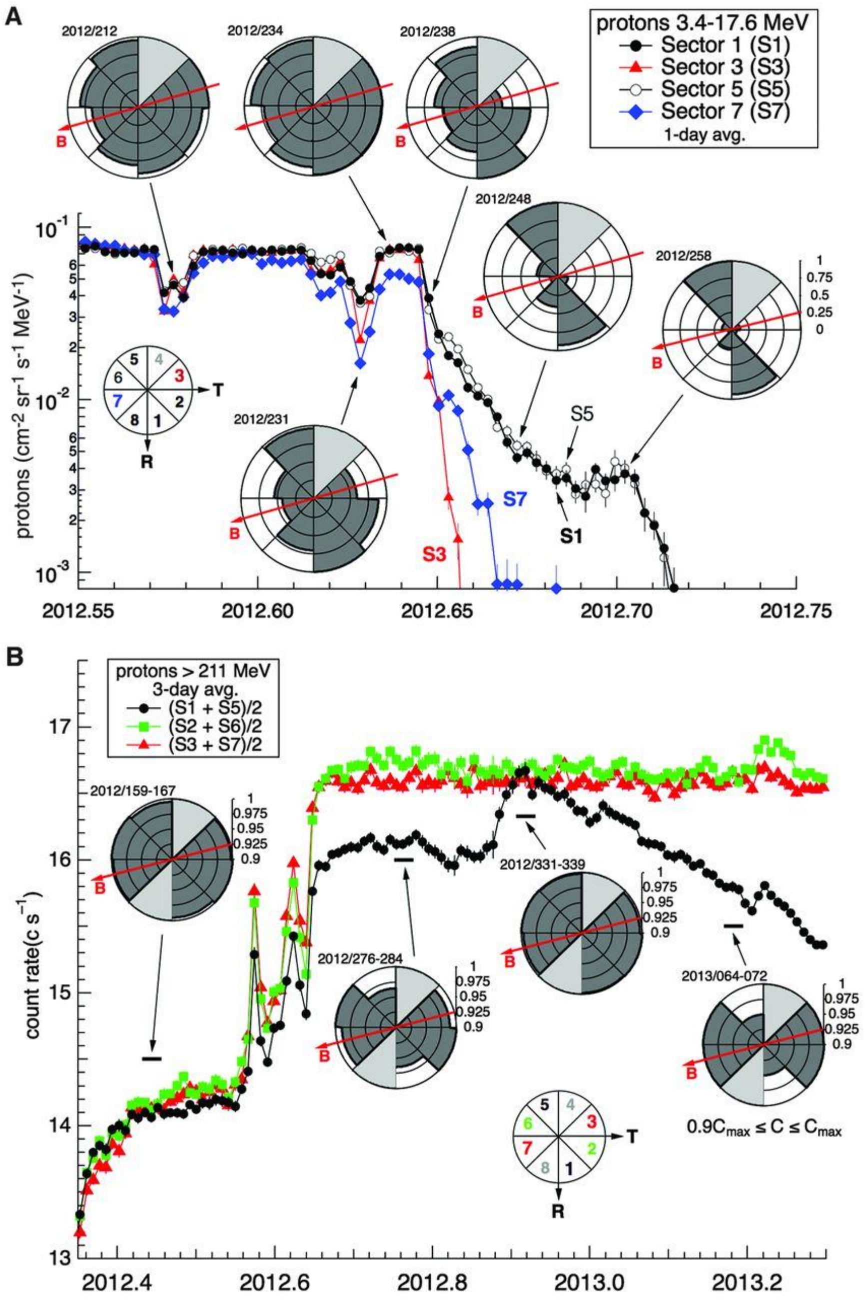


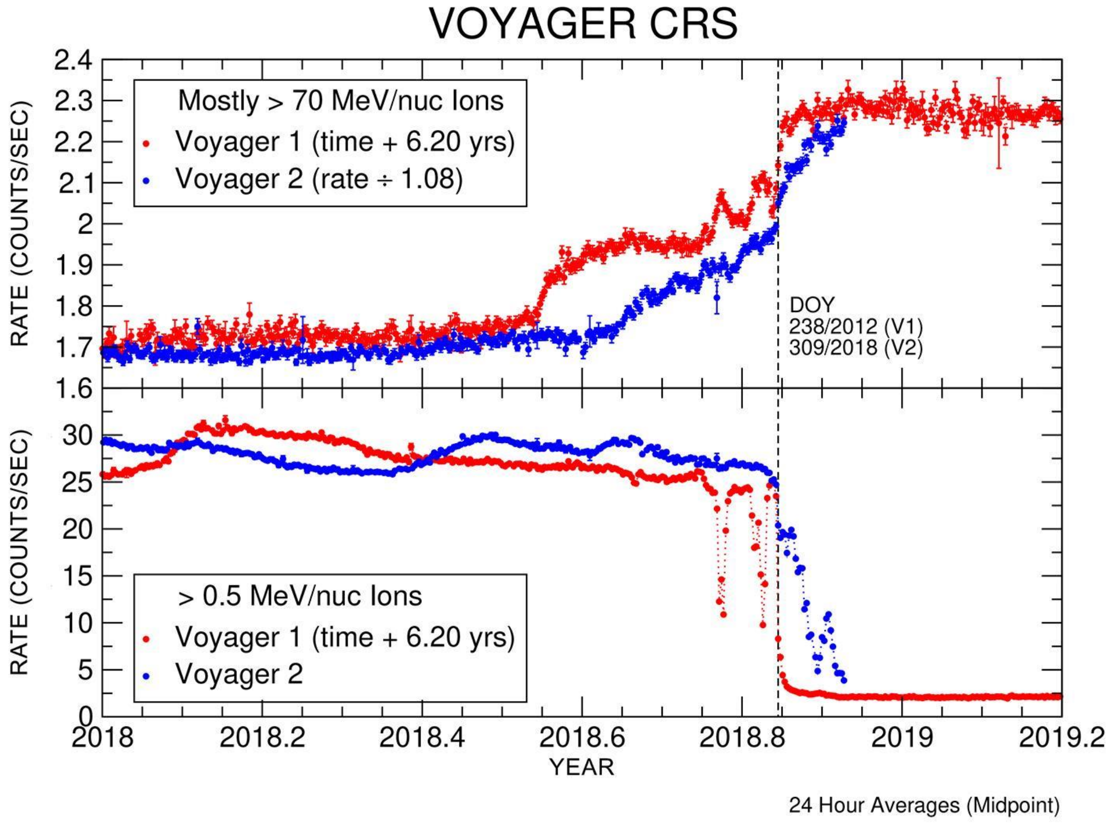

Highest energy ACRs are most intense at the flanks of the heliosphere

This is caused by the shock-shape (blunt) and magnetic field (nearly spherical) morphology

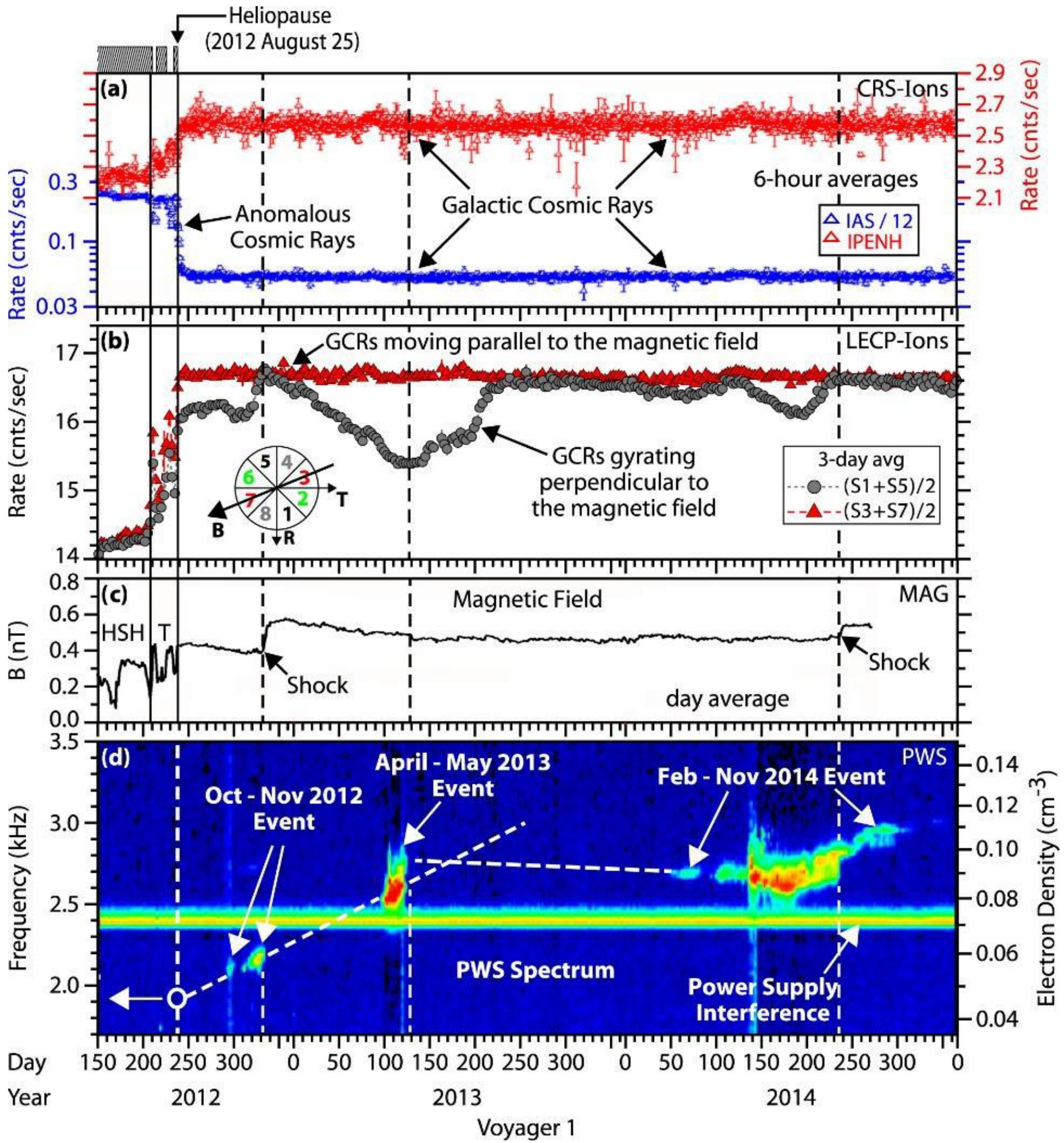
*McComas & Schwadron, 2006*

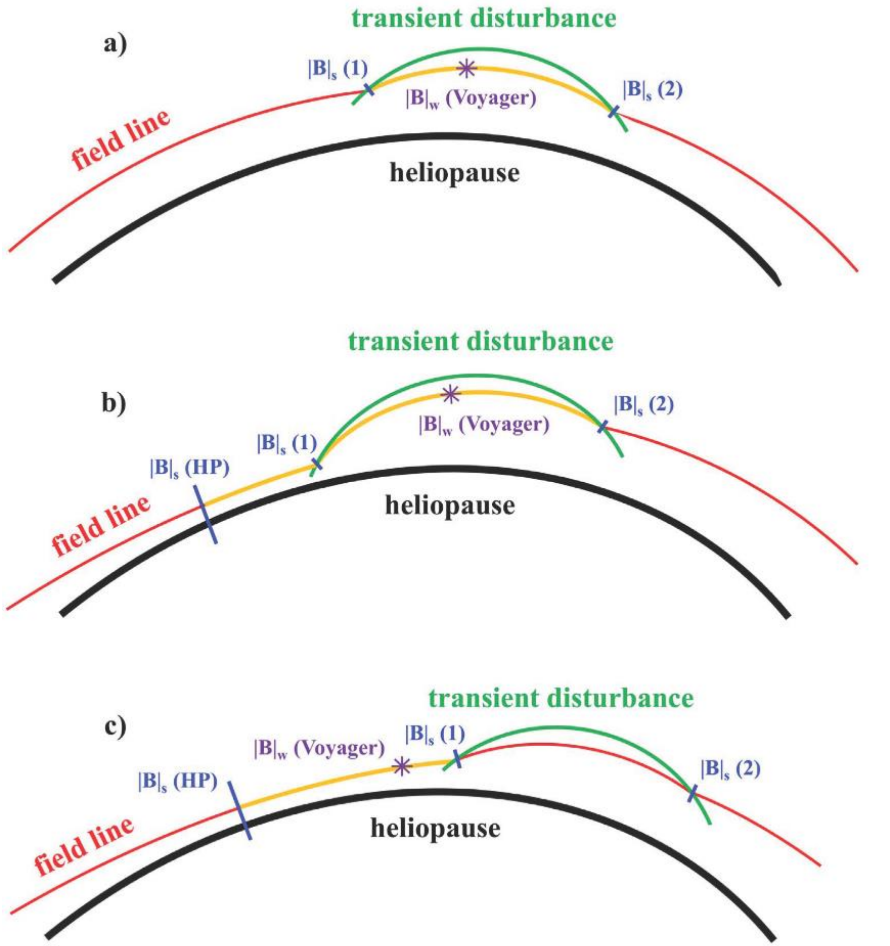
# GCRs and ACRs seen by the Voyagers (V1 and V2) as they crossed the heliopause into interstellar space



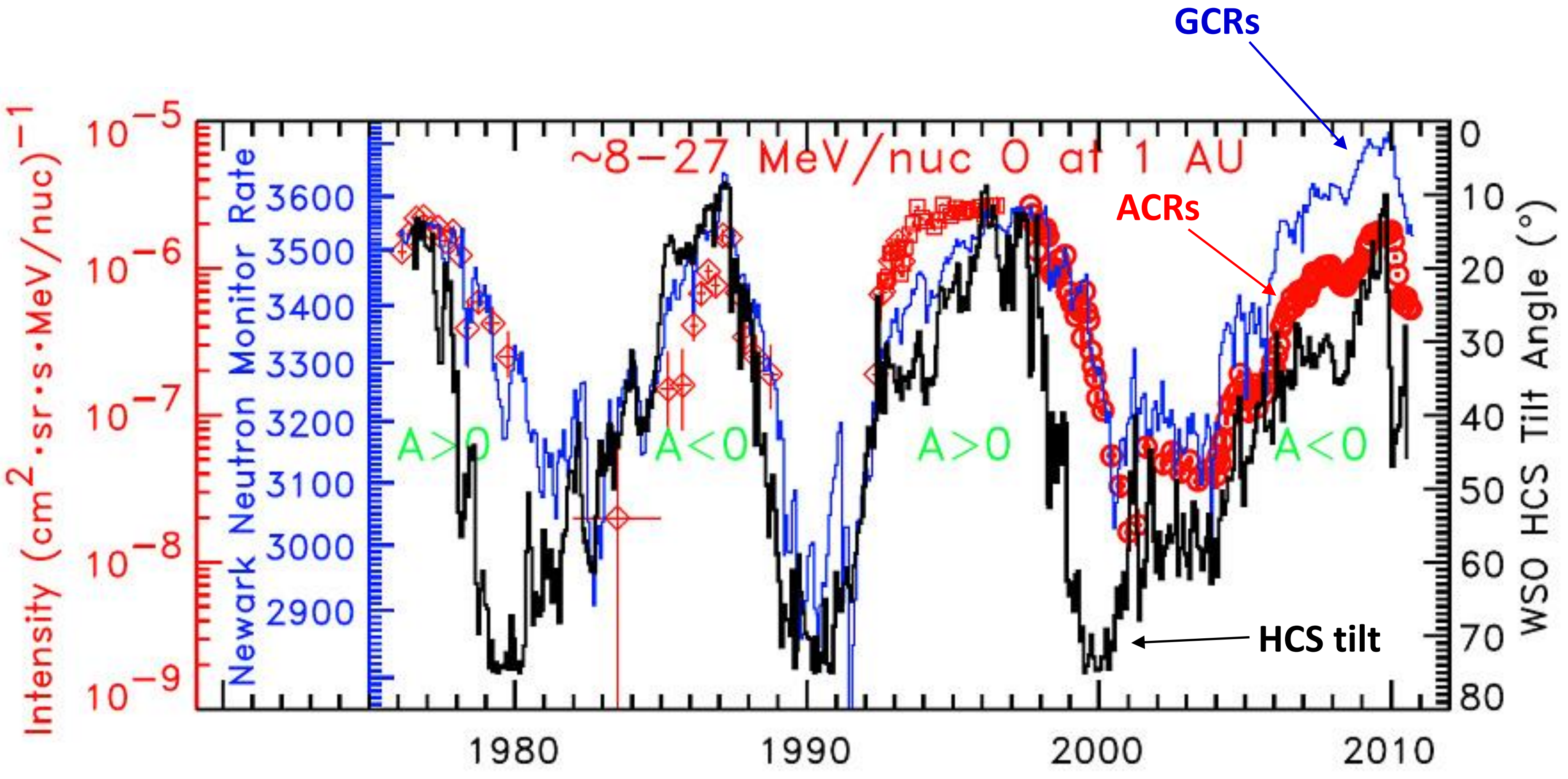

*Stone et al., Nature Astronomy, 2019*



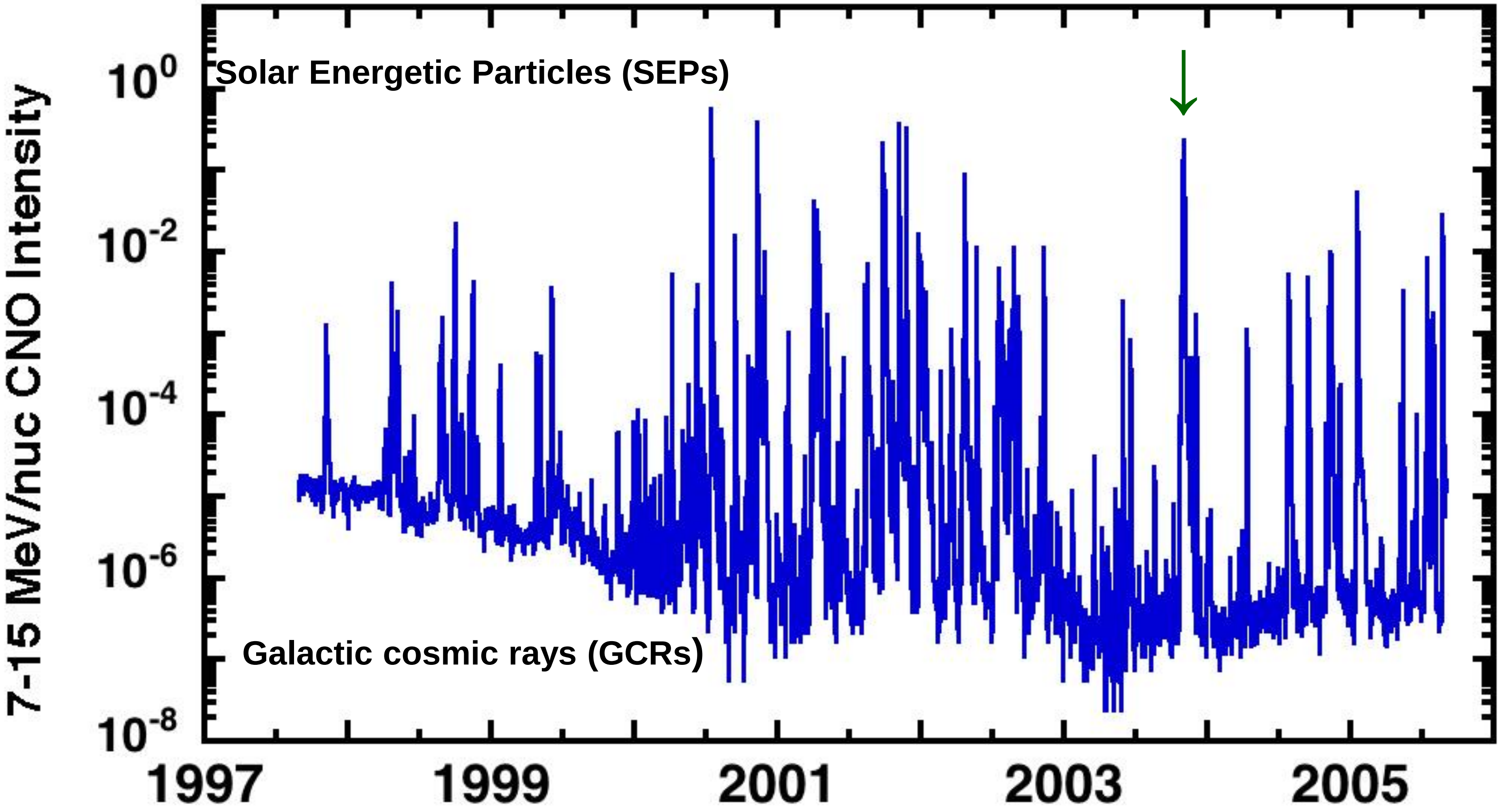



*Rankin et al., ApJ, 2019*

Although the GCR intensity in the last solar minimum was the highest ever during the space era … ACRs had a <u>*lower*</u> intensity compared to previous solar minima.

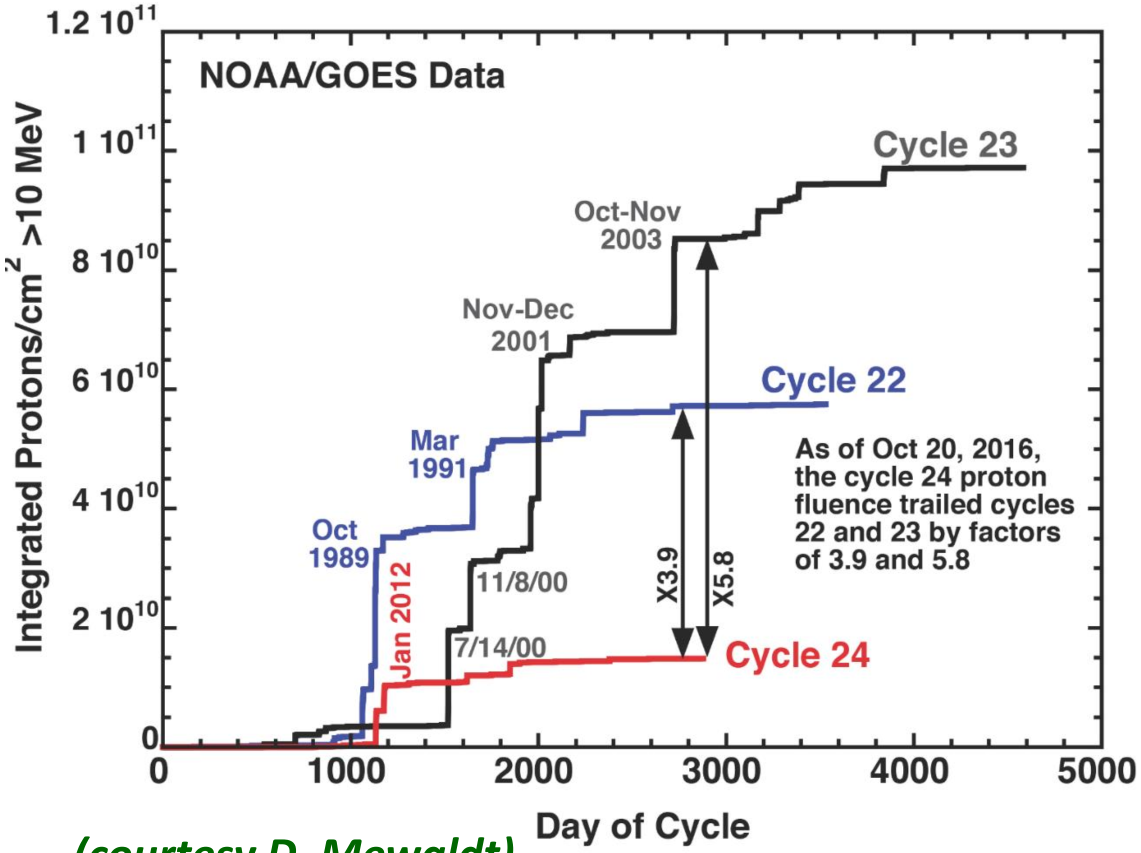


**Solar Energetic Particles vary unpredictably from one "event" to the next. The event indicated with the dark green arrow is a 1 million–fold increase in the intensity above the "background" GCR level**

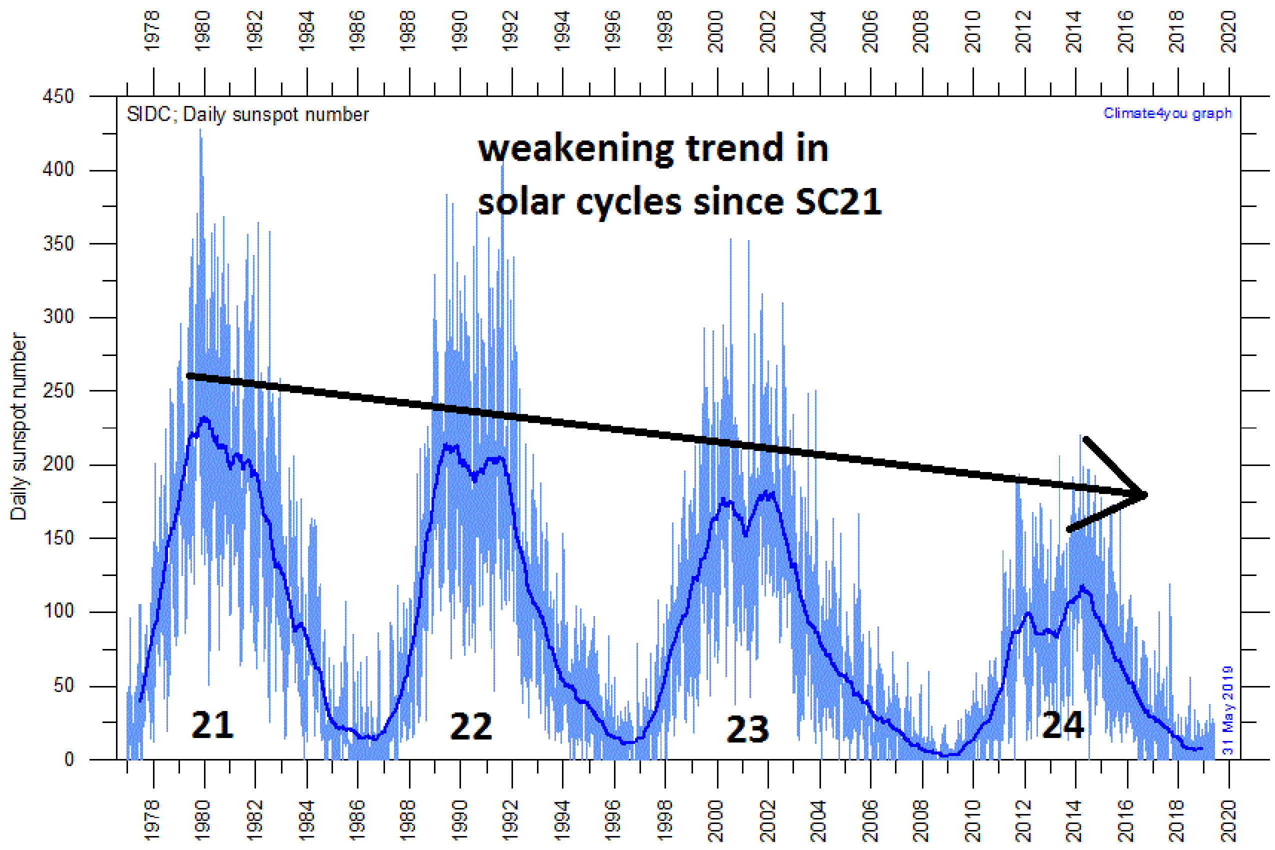


# There are far fewer SPEs in solar cycle 24 (most recent) compared to previous ones

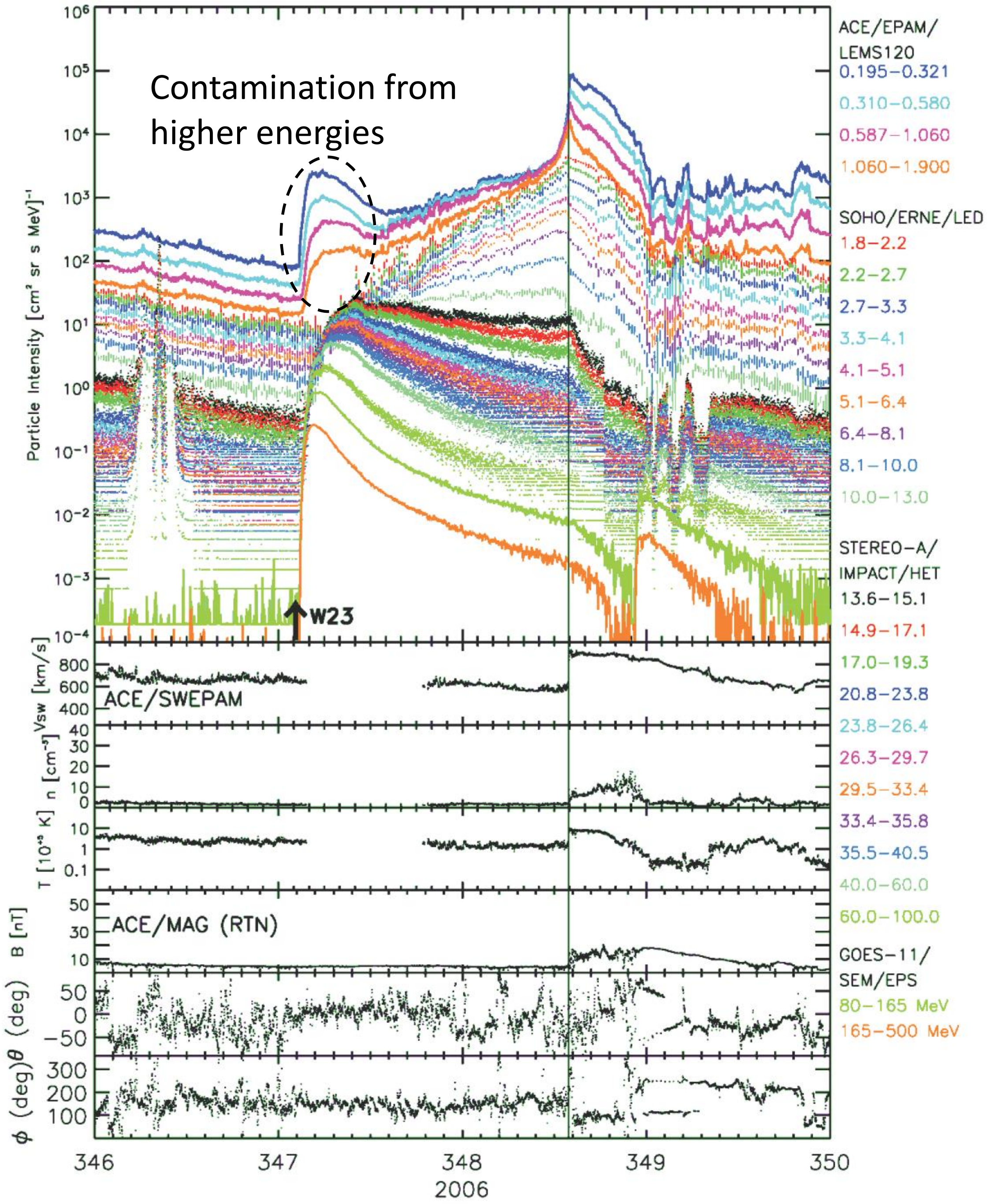
***Progression of Solar Proton Events For Solar Cycles 22, 23, and 24***



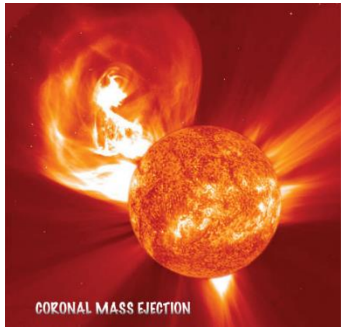

*(courtesy D. Mewaldt)*

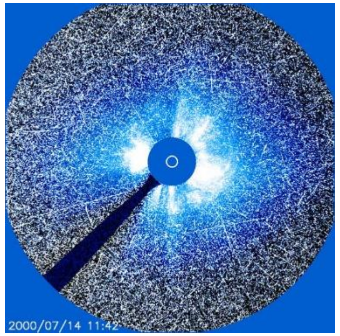


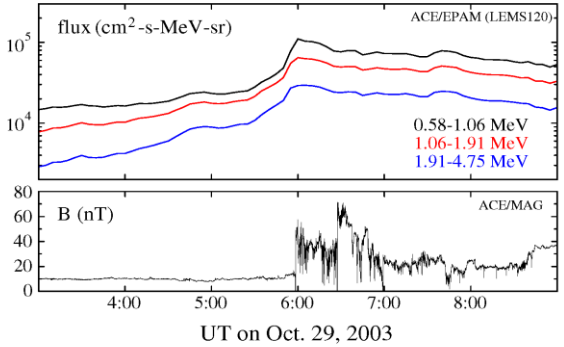
- A typical very large SEP event associated with a fast interplanetary (IP) shock seen by multiple spacecraft (ACE/SOHO/STEREO/GOES) near Earth
- The intensity vs. time depends on energy
- At low energies, the peak intensity is at the IP shock arriving ~1.5 days after the solar event; at high energies, the peak is well before the shock, closer to when the solar event was observed.
- The same shock likely accelerated these particles, but with a rate of acceleration that depends on the location of the shock




A sequence of events:



# The physics of high-energy charged particle transport (= propagation <u>and</u> acceleration).

- Treat as test particles.
- Assume there are no particle-particle collisions
- Equations of motion from the Lorentz force.
  - Need E and B fields.
    - Can get them from MHD simulations, for example, but this not "simple"
    - Can also get them from kinematic models. Easier, perhaps, but how realistic are they?
- Can also average over an ensemble of particles and arrive at very useful equations that describe the collective behavior of the particles, that are generally straightforward to implement on computers
  - Parker transport equation (Parker, 1965)
  - Focused-transport equation (Roelof, 1967; Ruffolo, 1995; Isenberg, 1997; Kota, 2000; Zhang et al., 2009; Droge et al., 2010)

# Particle transport in the heliosphere is actually the combination of four physical effects.

- ***Diffusion:*** caused by the scattering of the cosmic rays by the irregularities in the magnetic field. The associated "diffusion" is significantly larger along the magnetic field than normal to it.
- ***Convection:*** with the flow of the plasma.
- ***Guiding-center drifts:*** Such as gradient and curvature drifts, but also arising from interaction with current sheets in the solar wind
- ***Energy Change:*** caused by expansion/compression of the background fluid
  - such as a shock wave, which is a large plasma compression, leading to acceleration

**All of these effects play important roles in GCR, SEP, and ACR acceleration and modulation (it is often difficult to isolate the effects – they are all important)**

**These are combined in Parker's transport equation, first written down nearly 50 years ago.**

# The Parker Transport Equation (Parker, 1965):

$$\frac{\partial f}{\partial t} = -V_{w,i}\frac{\partial f}{\partial x_i} + \frac{\partial}{\partial x_i}\kappa_{ij}^{(S)}\frac{\partial f}{\partial x_j} - V_{D,i}\frac{\partial f}{\partial x_i} + \frac{1}{3}\frac{\partial V_{w,i}}{\partial x_i}\frac{\partial f}{\partial \ln p} + Q$$

***advection*** ***diffusion*** ***drift*** ***energy change*** ***source***

Where the drift velocity due to the large scale curvature and gradient of the average magnetic field is:

$$\mathbf{V_d} = \frac{pcw}{3q}\nabla \times \left[\frac{\mathbf{B}}{B^2}\right]$$

And the symmetric part of the diffusion tensor is:

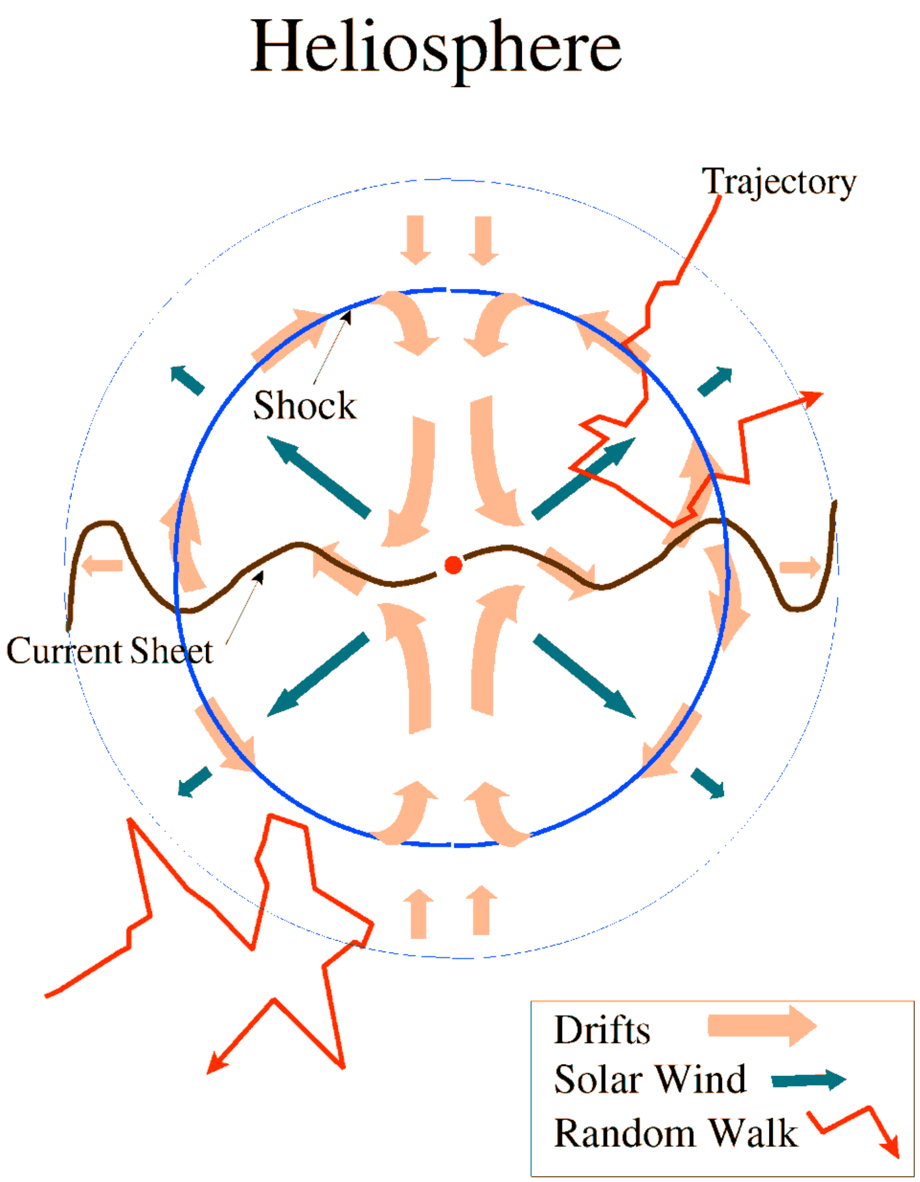
$$\kappa_{ij}^{(S)} = \kappa_\perp \delta_{ij} + (\kappa_\parallel - \kappa_\perp)\frac{B_i B_j}{B^2}$$

**The Parker Transport Equation is valid whenever the anisotropy is small (as observed for GCRs). It is widely used and remarkably general.**

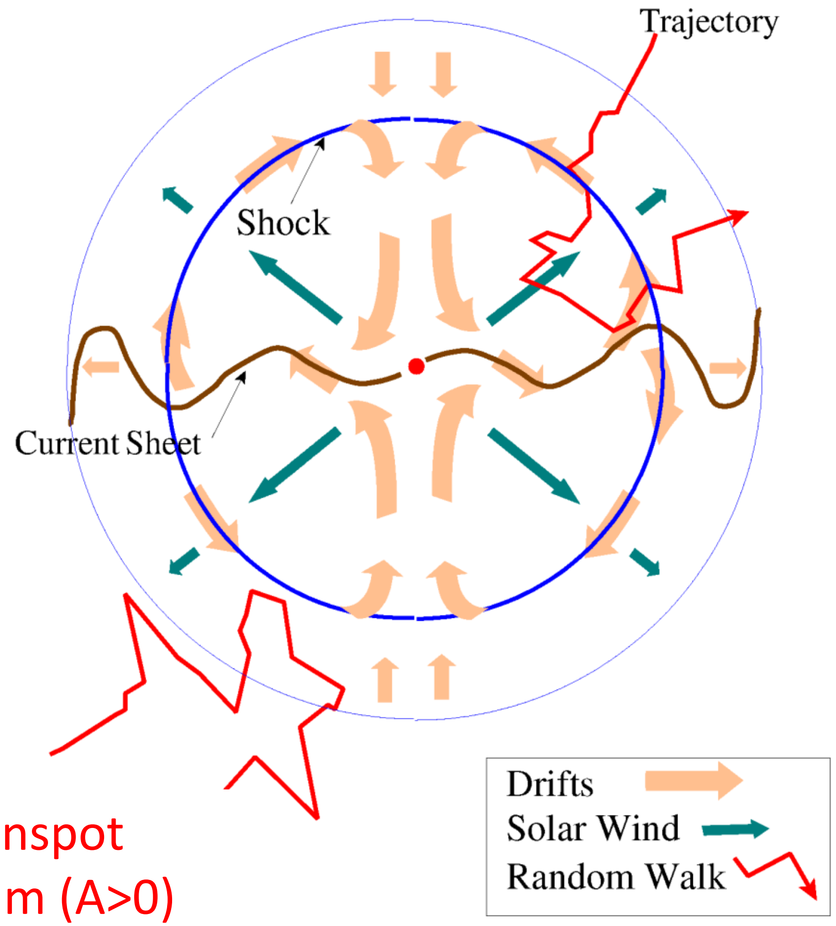
GCRs enter the heliosphere through a combination of diffusion and drift.

These motions are counteracted by outward convection and the associated cooling by the expansion of the wind.

Drift motions are very significant, and depend on the solar magnetic cycle. The pattern at right is for “A positive” (“A>0”, solar **B** is directed outward in northern solar hemisphere)

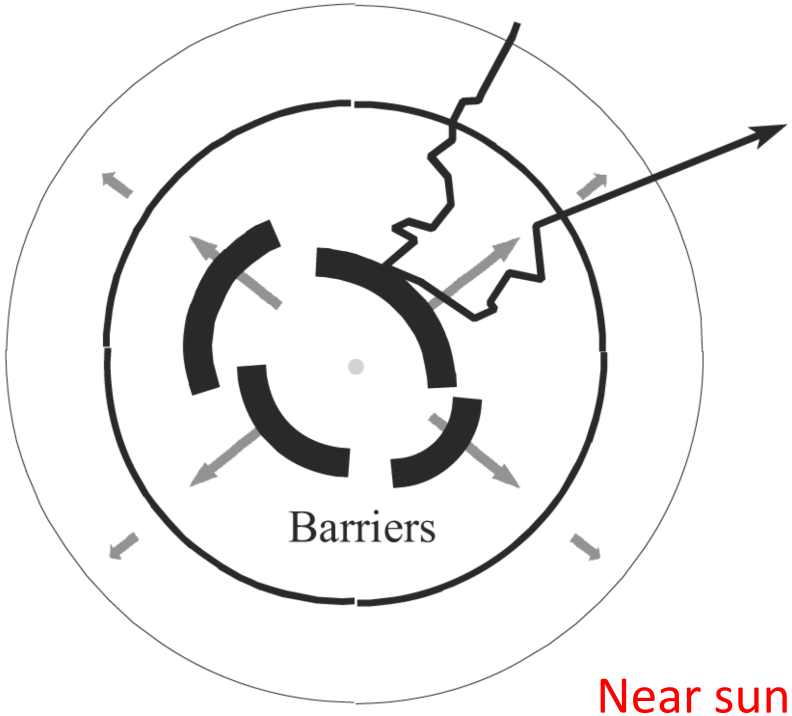


During ***solar minimum***, the interplanetary field is weaker, less solar-wind turbulence, fewer "barriers" (e.g. CMEs, shocks and merged interaction regions) and GCRs have easier access to 1 AU – ***higher GCR intensity***



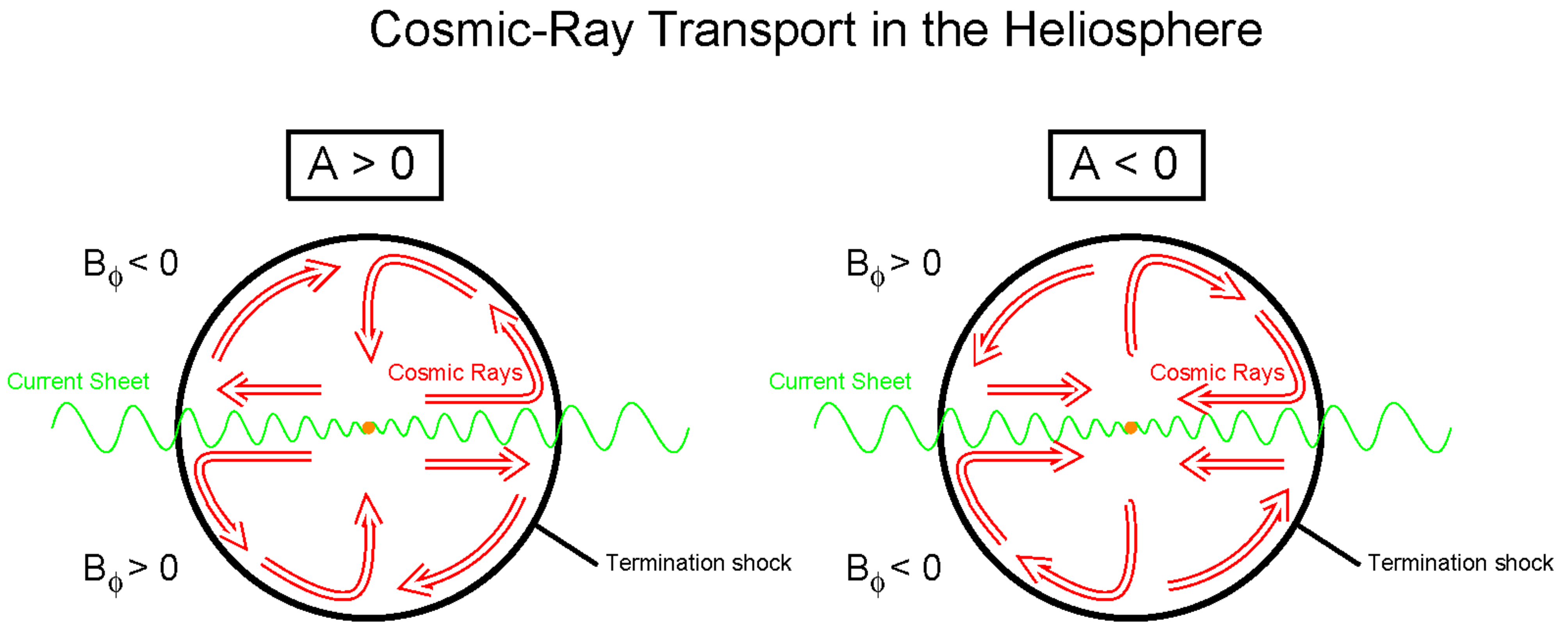

Near sunspot Minimum (A>0)

During ***solar maximum***, the interplanetary field is stronger, more turbulence, numerous large-scale barriers (e.g. CMEs), and GCRs have difficulty entering the solar system – ***lower GCR intensity***




Near sunspot Maximum

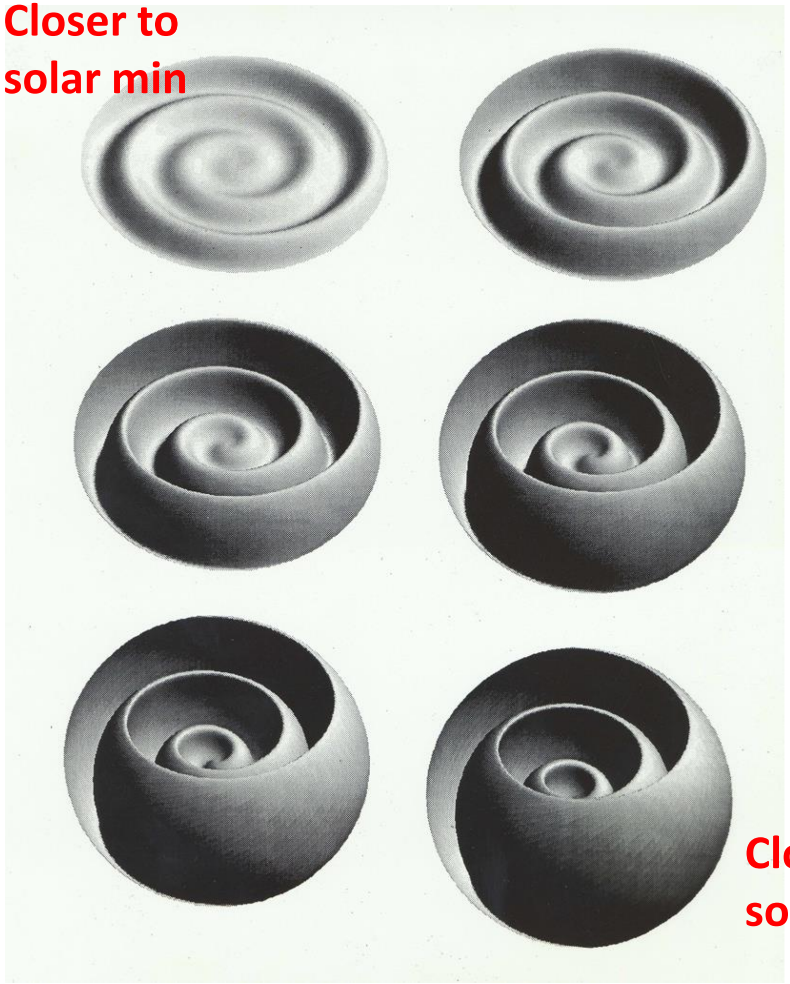
- The sense of the particle drift changes from one sunspot cycle to the next.
- For A>0, as GCRs enter the heliosphere, the drift brings them inward over the poles and out along the current sheet.
- The pattern is reversed for A<0 ("A negative")



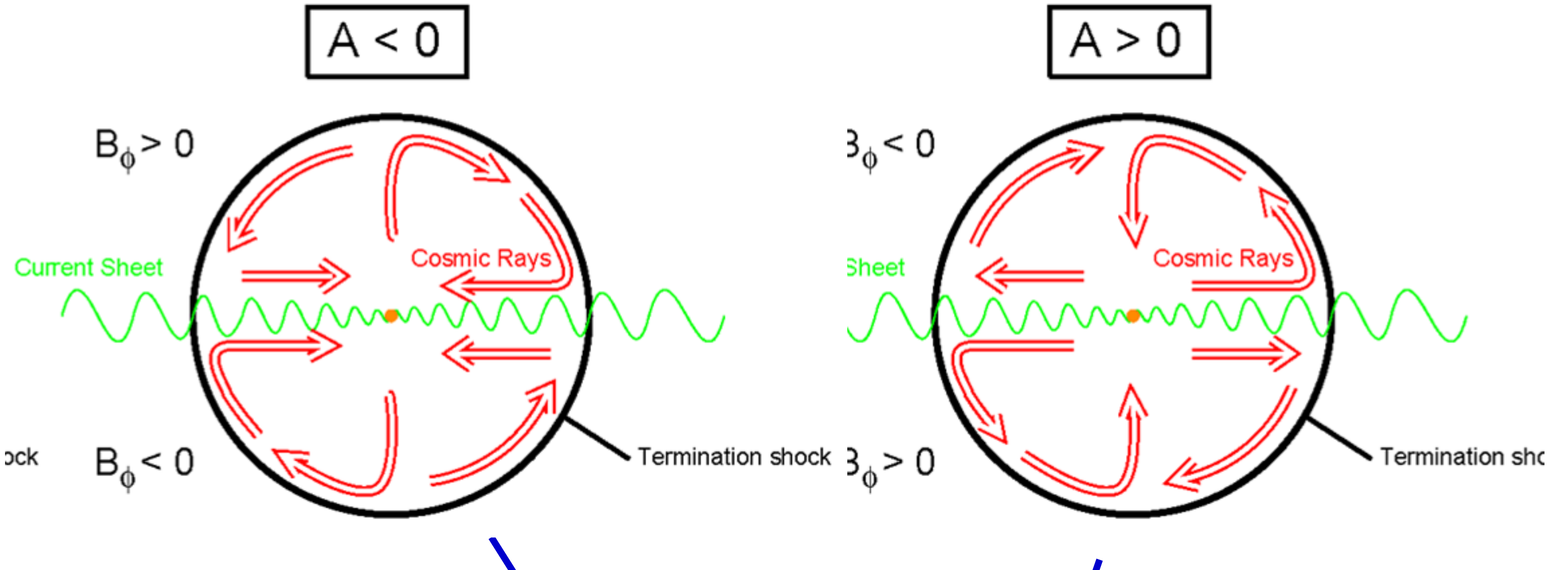

This is the current solar cycle

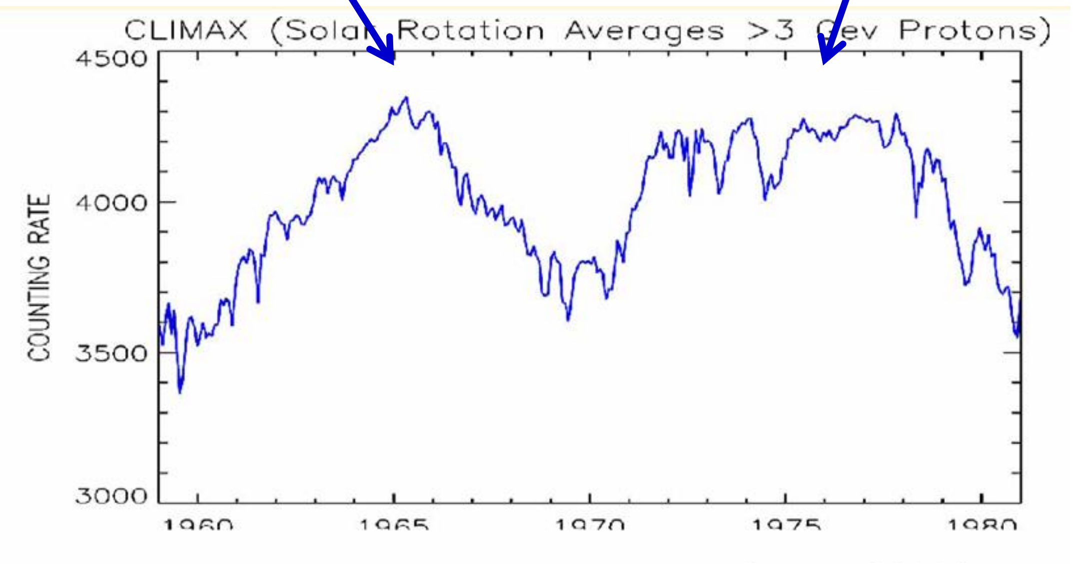
The current sheet changes from sunspot minimum to sunspot maximum. This effects the GCR intensity observed at 1AU

Closer to solar min



Closer to solar max



Generally, solving the Parker equation for the heliosphere requires a computer simulation. But, there are simple ways to get GCR modulation

- The Parker equation in spherical coordinates is:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\kappa_{rr}\frac{\partial f}{\partial r}\right) - U\frac{\partial f}{\partial r} + \frac{2Up}{3r}\frac{\partial f}{\partial p} = 0$$

- Which can be re-written:

$$\frac{\partial}{\partial r}r^2\left(\kappa_{rr}\frac{\partial f}{\partial r} - Uf\right) + \frac{2U}{3rp^2}\frac{\partial}{\partial p}(p^3 f) = 0$$

- If we neglect the last term (not a good approximation, by the way!), we get.

$$f(r,p) = f(R,p)\exp\left(-\int_r^R \frac{U}{\kappa_{rr}}dr'\right)$$

- The Parker equation can also be written in terms of the streaming flux, given by

$$\vec{S} = -\kappa\nabla f - \frac{\vec{U}}{3}\frac{\partial f}{\partial lnp}$$

- If assumed zero, and take $\kappa = g(r)wp$ it can be shown that

$$f(r,T) = f(R,T+\phi)$$

***"Force Field" Approximation***

- Where T is the kinetic energy, and ϕ is a potential of the form:

$$\phi = \int_r^R \frac{Uwp}{3\kappa_{rr}}dr'$$

***Modulation potential***

**In either/both of these simplified cases, it is clear that the modulation depends critically on the form of the diffusion coefficient**

# From the Lorentz-force on acting on individual charged particles to the Parker equation.

- Lorentz force (cgs, or “Gaussian” units, which I prefer)

$$\frac{d\vec{p}}{dt} = q\vec{E} + \frac{q}{c}\vec{w} \times \vec{B}$$

- Where $\boldsymbol{w} = \boldsymbol{p}/m$ is the particle velocity vector, and $\boldsymbol{p}$ is the momentum. $q$ is the particle’s charge, $\boldsymbol{E}$ and $\boldsymbol{B}$ are the electric and magnetic fields, respectively.
- Other forces are generally small, but can be added as needed (e.g. gravity, radiation pressure, etc.)

# Constant Electric and Magnetic Fields

Case A: $\boldsymbol{B}$ = constant

$\boldsymbol{E}$ = 0

$$\frac{d\vec{p}}{dt} = q\cancelto{0}{\vec{E}} + \frac{q}{c}\vec{w} \times \cancelto{B_0\hat{x}}{\vec{B}}$$

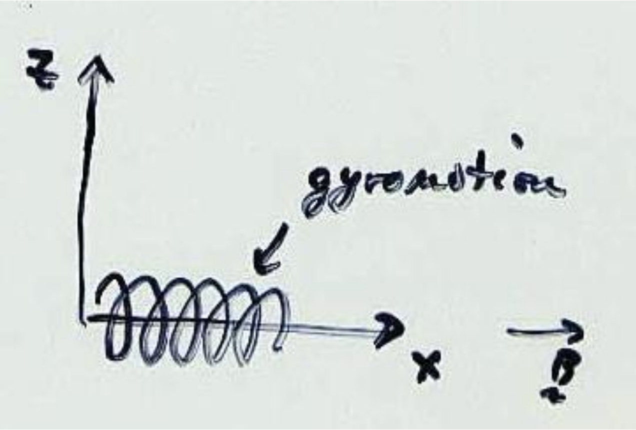
One gets simple gyromotion

The solution is given by:

$$w_x = w_\parallel = w\mu = const.$$
$$w_y = -w_\perp \cos(\Omega t - \phi)$$
$$w_z = w_\perp \sin(\Omega t - \phi)$$

where

$$\Omega = \frac{qB_0}{mc} \qquad w_\perp = \sqrt{w^2 - w_\parallel^2}$$
$$\phi = \tan^{-1}(w_{0z}/w_{0y})$$



# Constant Electric and Magnetic Fields Case

Case A: $\boldsymbol{B}$ = constant

$\boldsymbol{E}$ = constant

Assuming we have no parallel electric fields, we have

$$\vec{E} \cdot \vec{B} = 0$$

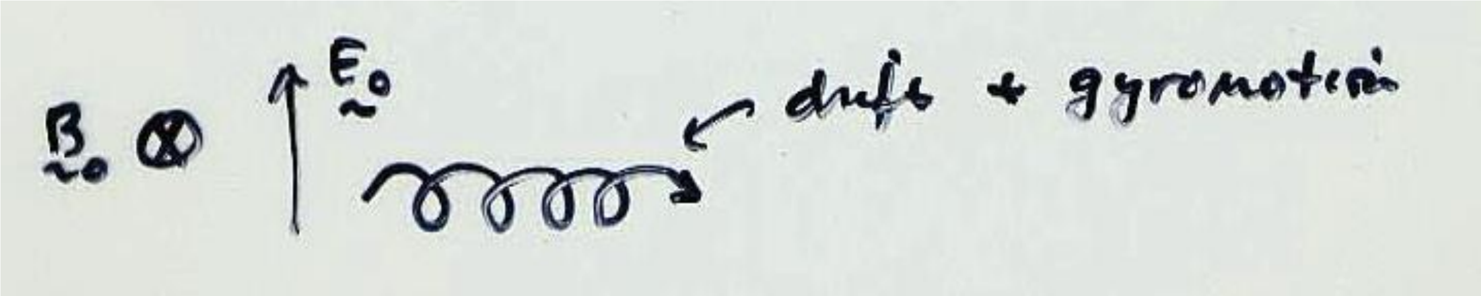
Then we find that in addition to gyromotion, there is also a drift such that

$$\vec{w} = \vec{w}_{gyro} + \vec{w}_{d,E}$$

where

$$\vec{w}_{d,E} = c\,\frac{\vec{E} \times \vec{B}}{\left|\vec{B}\right|^2}$$

Is called the electric-field drift



# Varying E and B fields
# Case 1: Scale of variation >> gyroradius of particles


$\mathbf{E} = -\frac{1}{c}\,\mathbf{u}\times\mathbf{B}$

if $r_g \ll L$

$\Rightarrow \mathbf{v} = \mathbf{v}_{gyromotion} + \mathbf{v}_{drift}$

Then other drifts enter, such as the "gradient" and "curvature" drifts, which, under the approximation of a curl-free magnetic field

$$\vec{v}_G = \frac{cW_\perp}{qB^3}\vec{B} \times \nabla B$$

$$\vec{v}_C = \frac{2cW_\parallel}{qB^3}\vec{B} \times \nabla B$$

$W_\perp$ , $W_\parallel$ are the components of the particle's kinetic energy perpendicular and parallel to the average magnetic field

The general expression for motion of the center of gyration of the particle about the field is given by (***b*** is a unit vector along ***B***)

$$\vec{v}_{g.c.} = \left[w_\parallel + \frac{cW_\perp}{2qB}\hat{b}\cdot\left(\nabla \times \hat{b}\right)\right]\hat{b} + \frac{cW_\perp}{2qB^2}\hat{b} \times \nabla B + \frac{cW_\parallel}{qB}\hat{b} \times \left(\hat{b}\cdot\nabla\right)\hat{b}$$

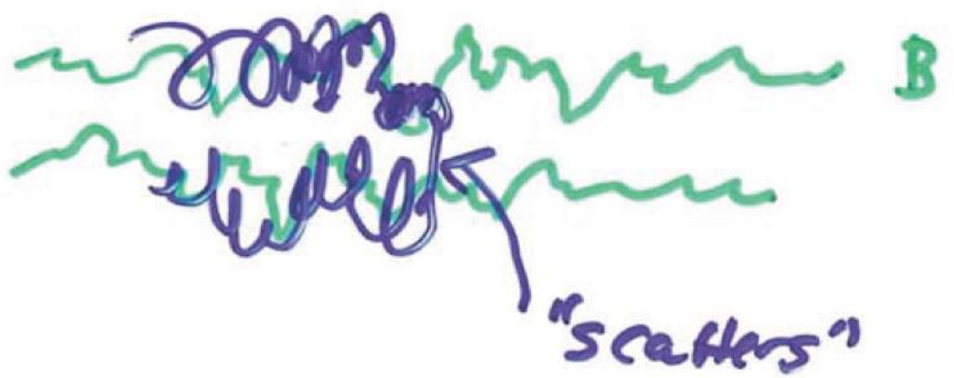
That, when averaged over an isotropic distribution of particles gives:

$$\vec{V}_d = \frac{cmw^2}{q}\nabla \times \left(\frac{\vec{B}}{B^2}\right)$$

**The text book by Rossi & Olbert "Introduction to the Physics of Space" (1970) is an excellent resource for this topic.**

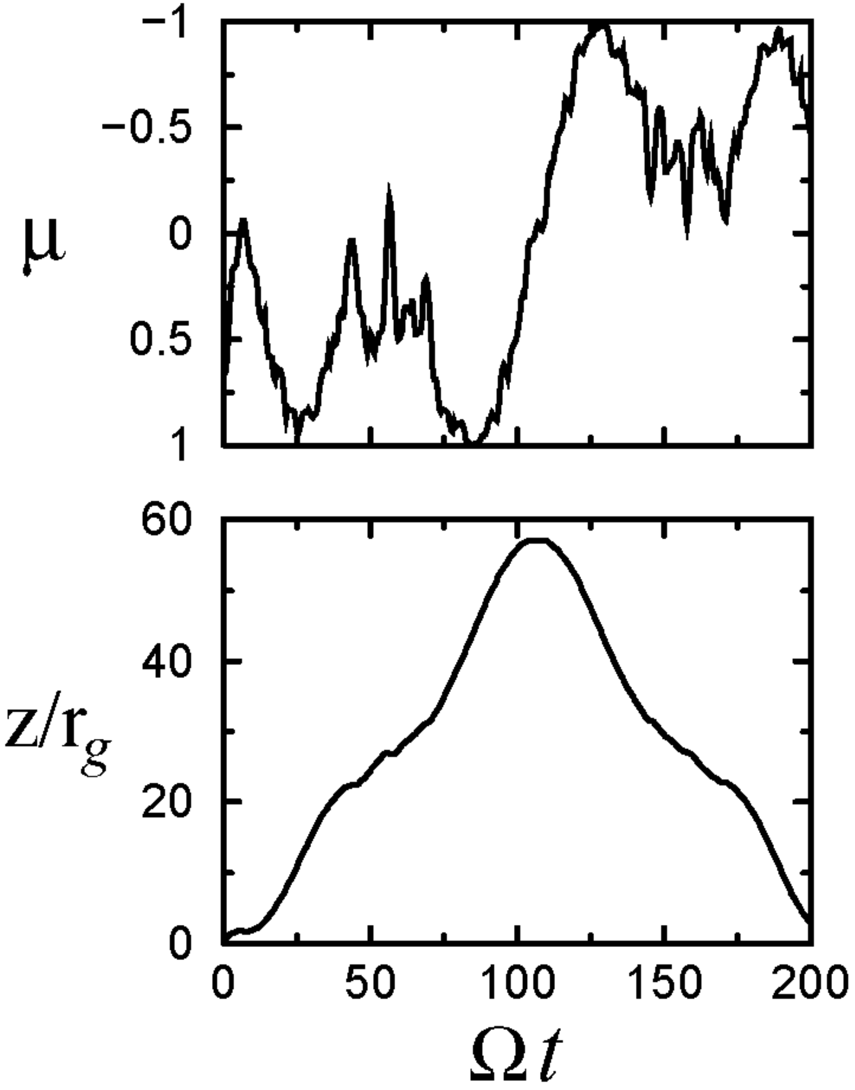
# Varying E and B fields
# Case 2: Scale of variation ≈ gyroradius of particles

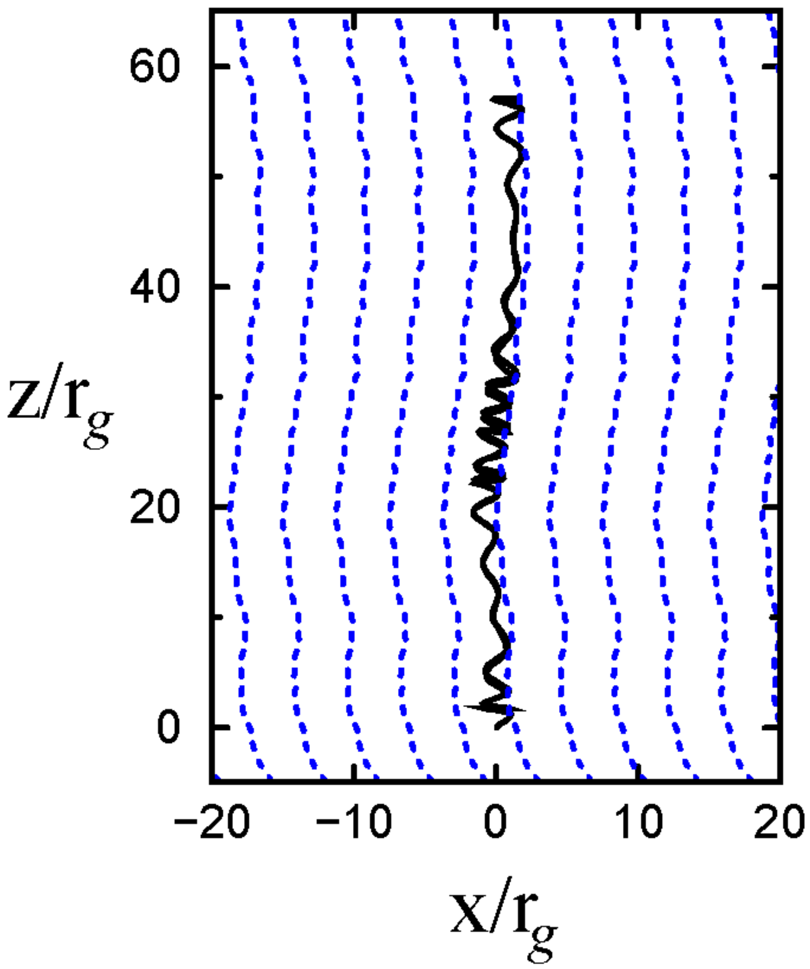
what if $r_g \sim L$ ?




- A "resonance" can occur such that the particles pitch angle is reversed. This is much like a "scattering" event in scattering theory. The resonance condition is $kw\mu = \Omega$, where $k$ is the wavenumber of the fluctuation, $w$ is the particle speed, $\mu$ is the cosine of the pitch angle, and $\Omega$ is the particle cyclotron frequency.

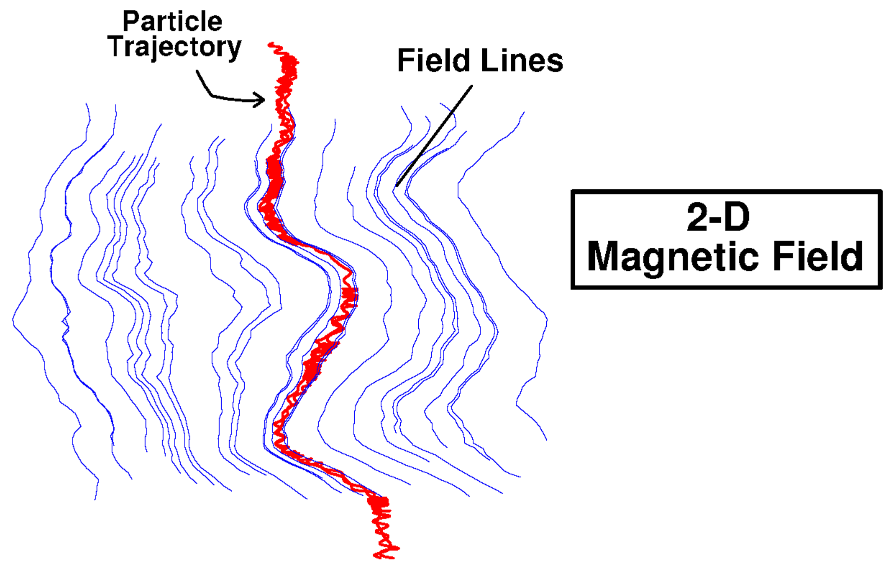
# A charged particle moving in a turbulent magnetic field (numerical integration)

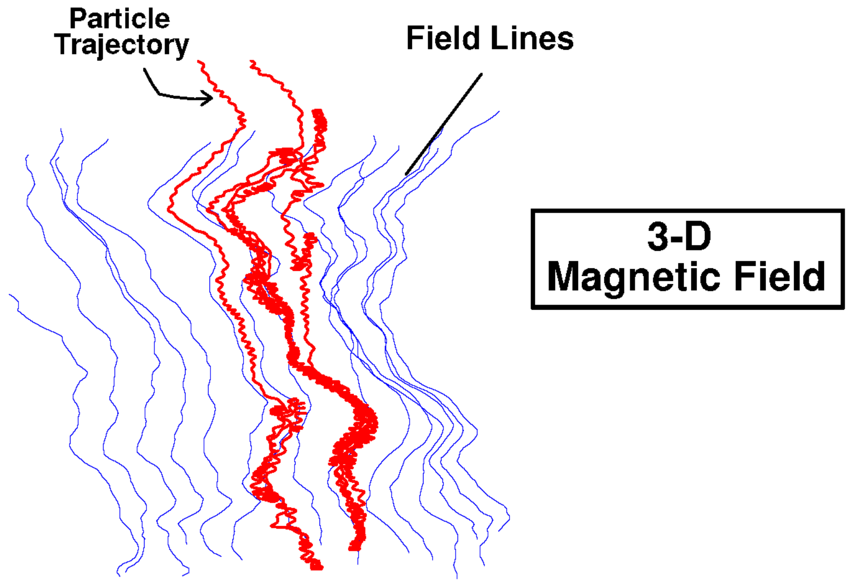


# Restrictions on particle motions imposed by artificially limiting the dimensionality of the fields

- Charged particles are strictly tied to magnetic lines of force in 1 and 2D electric and magnetic fields
  - This can be proven rigorously and follows directly from the equations of motion
- This is an artificial and unphysical constraint on charged-particle motion!
  - Be aware!



# Pitch-angle , spatial diffusion, and the quasi-linear theory

- We assume charged particles undergo pitch-angle diffusion and their distribution satisfies

$$\frac{df(\mu,z,t)}{dt} = -w\mu\frac{\partial f(\mu,z,t)}{\partial z} + \frac{\partial}{\partial \mu}\left(D_{\mu\mu}(\mu)\frac{\partial f(\mu,z,t)}{\partial \mu}\right)$$

***This comes from the Boltzmann equation***

- $D_{\mu\mu}$ is the pitch-angle diffusion coefficient, μ is the pitch cosine. w is the particle speed, z is the direction along the mean magnetic field.
- By assuming small anisotropy, this can be written as a spatial diffusion equation:

$$\frac{df(z,t)}{dt} = \frac{\partial}{\partial z}\left(\kappa_{\parallel}\frac{\partial f(z,t)}{\partial z}\right)$$

***Jokipii, (1966, 1969)***
***Hasselmann & Wibberenz, (1970)***
***Earl, (1974)***
***Luhmann, (1978)***

- Where $\kappa_{\parallel}$, is related to the $D_{\mu\mu}$ by

$$\kappa_{\parallel}(w) = \frac{w^2}{4}\int_0^1 \frac{(1-\mu^2)^2}{D_{\mu\mu}}d\mu$$

Mean-free path $\lambda_{\parallel} = 3\kappa_{\parallel}/w$

# Pitch-angle , spatial diffusion, and the quasi-linear theory (cont.)

- The “scattering” is caused by turbulent magnetic fields. $D_{\mu\mu}$ is obtained by integrating the equations of motion of particles moving within these fields (*Jokipii*, 1966). For a mean magnetic field along z, the relevant equations are:

$$\frac{dp_z}{dt} = \frac{q}{c}\left(v_x \delta B_y - v_y \delta B_x\right) \qquad \mu = {}^{p_z}/_{p} \qquad p = const.$$

- Solved by inserting the zeroth order (solution to equations of motion for no fluctuating field) into the above, dropping terms second order in perturbed quantities, integrating over time, averaging over gyro-phase and field realizations. One obtains the well-known result:

$$D_{\mu\mu} = \lim_{\Delta t \to \infty} \frac{\langle (\Delta\mu)^2 \rangle}{2\Delta t} = \frac{\pi}{4}\Omega_0 (1-\mu^2)\frac{k_r P(k_r)}{B_0^2}$$

***Jokipii, (1966)***

# Pitch-angle , spatial diffusion, and the quasi-linear theory (cont.)

- Assuming a power spectrum of the form:

$$P(k) = \frac{C}{1 + (kL_c)^{5/3}} \qquad \sigma^2 = \int_0^\infty P(k)dk$$

- $L_c$ is the turbulence correlation length. We obtain

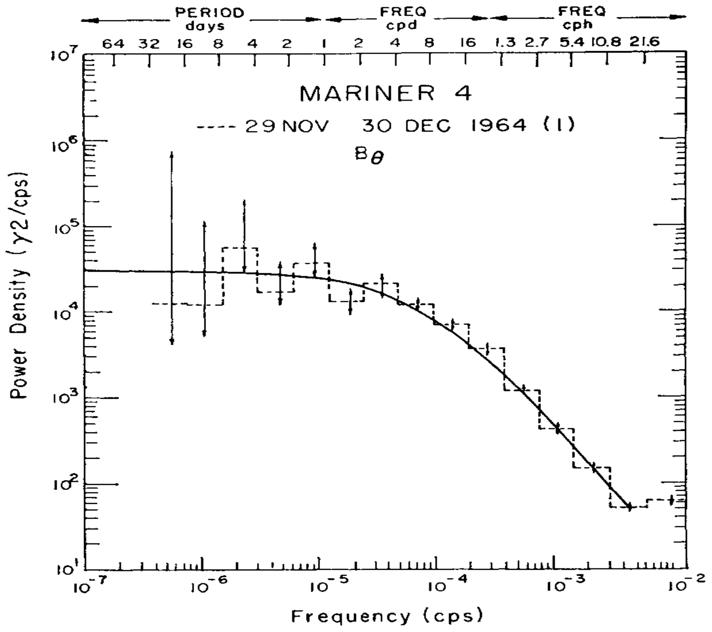
$$\kappa_\parallel = \frac{3v^2}{20L_c\Omega_0^2}\left(\frac{B_0}{\sigma}\right)^2 \csc(3\pi/5)\left[1 + \frac{72}{7}\left(\frac{\Omega_0 L_c}{v}\right)^{5/3}\right]$$

- Can also solve for cross-field transport

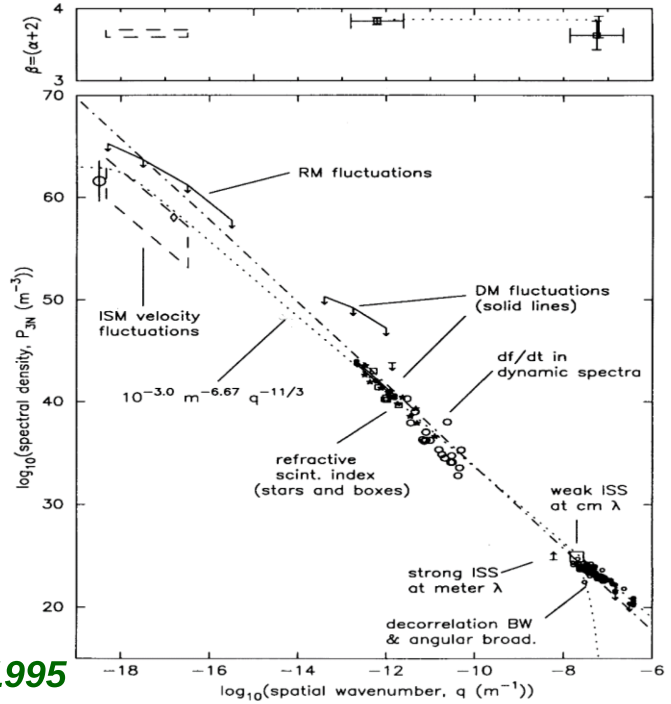
$$\kappa_\perp = \frac{5vL_c}{12}\left(\frac{\sigma}{B_0}\right)^2 \sin(3\pi/5)$$

*Jokipii & Coleman, JGR, 1968*

Interplanetary turbulence




Interstellar turbulence




*Armstrong et al, 1995*

In addition to diffusion, there are other important collective effects on charged particles: they include

- Advection

$$\vec{U} \cdot \nabla f$$

(arises because the "scattering centers" are moving with the bulk plasma flow)

- Energy Change

$$\frac{p}{3} \nabla \cdot \vec{U} \frac{\partial f}{\partial p}$$

(arises because of scattering in converging or diverging flows)

- Note that the energy-change term does not contain the electric field! In fact, the electric field is what is responsible for energy changes.
- Starting with the work done on a particle by *ideal MHD* electric field:

$$\frac{d}{dt}\left(\frac{p^2}{2m}\right) = -q\vec{w} \cdot \vec{E} = \frac{q}{c}\vec{w} \cdot \left(\vec{U} \times \vec{B}\right)$$

- Then, by using a vector identity, and averaging over an isotropic distribution of particles, it follows (c.f. Jokipii, AIP Conf. Proc., 2012)

$$\frac{dp'}{dt} = -\frac{p'}{3}\nabla \cdot \vec{U}$$

- Which appears in the energy-change term, where p’ is the momentum in the frame of reference moving with ***U***.

# Additional hand-written notes for additional reference

- These notes originated from in-class lectures in the spring of 2018 (and again in spring 2020, virtually) at the University of Arizona in the graduate level course: “Plasma Physics with Applications to Astrophysics and Solar Physics”

- Included here are most steps to the derivations of (a) the spatial diffusion coefficient  in terms of the pitch-angle diffusion coefficient, and (b) the energy-change term.

## Charged-particle transport : pitch-angle diffusion, spatial diffusion (parallel transport)

Consider a turbulent magnetic field of the form

$$\underset{\sim}{B} = B_0 \hat{z} + \delta B_x(z,t)\,\hat{x} + \delta B_y(z,t)\,\hat{y}$$

generally, $\underset{\sim}{\delta b} << \underset{\sim}{B}_0$

these are Alfven waves, $\underset{\sim}{k} = k\hat{z}$

Consider the equation of motion

$$\frac{dv_x}{dt} = \frac{q}{mc}\left(v_y B_0 - v_z \delta B_y\right)$$

$$\frac{dv_y}{dt} = \frac{q}{mc}\left(-v_x B_0 + v_z \delta B_x\right)$$

$\delta B << B_0$ $\Rightarrow$ just give the usual oscillatory motion

$$\frac{dv_z}{dt} = \frac{q}{mc}\left(v_x \delta B_y - v_y \delta B_x\right)$$

the last eq. is of interest for us (parallel transport)

re-write $v_z = v\mu$ $\quad$ $\mu$ = pitch cosine

$v$ = constant if $\underset{\sim}{\delta B} \neq \underset{\sim}{\delta B}(t)$

(if $v >> v_{phase}$ of waves, the $\underset{\sim}{\delta B}$ will appear static)

$\Rightarrow \delta B$ is time-independent

~~we~~ because $\delta \underset{\sim}{B} \ll \underset{\sim}{B}_0$, we can substitute the zeroth order solution for $v_x$ & $v_y$.

$$\Rightarrow \quad v\frac{d\mu}{dt} = \frac{q}{mc}\Big(\underbrace{v_\perp \cos(\Omega_0 t - \phi)}_{v_x}\, \delta B_y(v\mu t) + \underbrace{v_\perp \sin(\Omega_0 t - \phi)}_{-v_y}\, \delta B_x(v\mu t)\Big)$$

$$\frac{d\mu}{dt} = \frac{q v_\perp}{vmc}[\;] \qquad\qquad \Omega_0 = \frac{qB_0}{mc}$$

$$= \frac{q(1-\mu_0^2)^{1/2}}{mc}[\;]$$

$$= \frac{q(1-\mu_0^2)^{1/2}}{mc}\Big[\cos(\Omega_0 t - \phi)\,\delta B_y(v\mu_0 t) + \sin(\Omega_0 t - \phi)\,\delta B_x(v\mu_0 t)\Big]$$

$\delta B_x, \delta B_y \rightarrow$ sinusoidal functions

$$\sim e^{ikz} \sim e^{ikv\mu_0 t}$$

Solution is

$$\mu = \int_0^t dt' \frac{q(1-\mu_0^2)^{1/2}}{mc}[\;](t')$$

resonances will occur when $kv\mu_0 \sim \Omega_0$

gyroradius ~ wavelength

Jokipii, 1966 noted

$$\langle \mu^2 \rangle = \int_0^t dt' \int_0^t dt'' \frac{q^2(1-\mu_0^2)}{m^2c^2} \langle [\;](t')[\;](t'') \rangle$$

the part in $\langle\ \rangle$ on the right ~~terms~~ leads to a diffusive process, that is

$\langle \mu^2 \rangle \propto t \Rightarrow$ diffusive

proportionality depend on Power spectrum associated with $\delta \underset{\sim}{B}$ fluctuations

define pitch-angle diffusion coefficient, $D_{\mu\mu}$

$$\boxed{D_{\mu\mu} = \frac{\langle \Delta\mu^2 \rangle}{2\Delta t} = \frac{\pi}{4}(1-\mu^2)\,\Omega_0\,\frac{k_r\,P(k_r)}{B_0^2}}$$

Jokipii, 1966

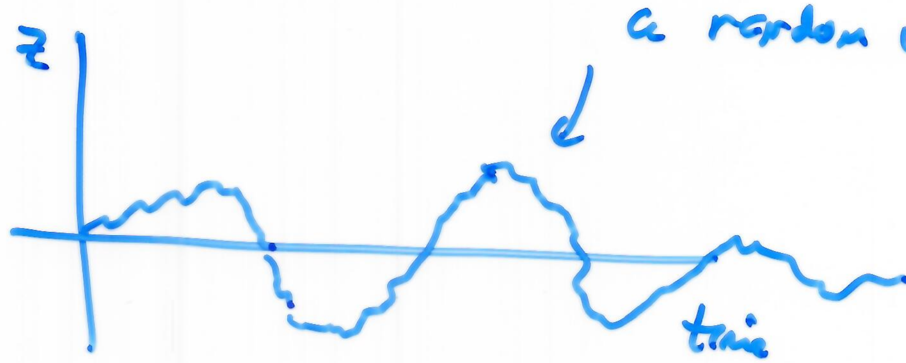
$P(k) =$ Power spectrum of mag. field

$P = P_{xx} = P_{yy}$ in this case

$k_r = \left| \frac{\Omega}{v\mu} \right| \rightarrow$ resonant wave number

What about spatial diffusion?

Consider




$$\frac{dz(t)}{dt} = v_z(t)$$

$$\rightarrow \quad z(t) = \int_0^t dt'\, v_z(t') \qquad \langle z \rangle = 0$$

$$z^2(t) = \int_0^t \int_0^t dt'\, dt''\, v_z(t')\, v_z(t'')$$

$$\langle z^2 \rangle = \int_0^t \int_0^t dt'\, dt'' \langle v_z(t')\, v_z(t'') \rangle$$

let $t'' = t' + \tau$

$$\langle z^2 \rangle = \int_0^t \int_{-t'}^{t-t'} dt'\, d\tau\, \underbrace{\langle v_z(t')\, v_z(t'+\tau) \rangle}_{R(\tau)\ \rightarrow\ \text{does not depend on } t'}$$

$$= \int_0^t \int_{-t'}^{t-t'} dt'\, d\tau\, R(\tau)$$

without loss of generality is 

$$\langle z^2 \rangle = \int_0^t \int_{-\infty}^{\infty} dt' \, d\tau \, R(\tau)$$

$$= t \int_{-\infty}^{\infty} R(\tau) \, d\tau$$

$$R(\tau) = R(-\tau)$$

$$\Rightarrow \langle z^2 \rangle = 2t \int_0^{\infty} R(\tau) \, d\tau$$

$\Rightarrow \dfrac{\langle z^2 \rangle}{2t} \neq$ function of time $\Rightarrow \langle \Delta z^2 \rangle \propto t$

$\Rightarrow$ diffusion

this is spatial diffusion w/ coefficient K

$$K = \frac{\langle \Delta z^2 \rangle}{2 \Delta t} = \frac{1}{3} \lambda v \qquad \lambda = \text{mean-free path}$$

A distribution of particles undergoing diffusion obeys

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial z}\left(K \frac{\partial f}{\partial z}\right) \qquad \text{Diffusion equation}$$

So, how do we relate $K$ to $D_{\mu\mu}$? why do this? because I want to know how $K$ is related to magnetic field power spectrum.

Consider the Boltzmann eq. along the $z$ direction

$$\frac{\partial f}{\partial t} + v\mu \frac{\partial f}{\partial z} + \underbrace{\left(\frac{\underset{\sim}{F}}{m} \cdot \nabla_{\underset{\sim}{v}} f\right)_z}_{\text{do quasi-linear theory (Jokipii, 1966)}} = \left(\frac{\partial f}{\partial t}\right)_c$$

we find, $\left(\left(\frac{\partial f}{\partial t}\right)_c = 0\right)$ strictly particle-particle collisions

⋮

$$\frac{\partial f}{\partial t} + v\mu \frac{\partial f}{\partial z} = \frac{\partial}{\partial \mu}\left(D_{\mu\mu} \frac{\partial f}{\partial \mu}\right)$$

pitch-angle diffusion equation

we also have

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial z}\left(K \frac{\partial f_0}{\partial z}\right), \text{ where } f_0 = \int_{-1}^{1} f \, d\mu$$

## Particle Transport (cont.)

Pitch angle diffusion & its relation to spatial diffusion

recall, we had.

$$\frac{\partial f}{\partial t} + v\mu \frac{\partial f}{\partial z} = \frac{\partial}{\partial \mu}\left(D_{\mu\mu} \frac{\partial f}{\partial \mu}\right)$$

assume

$$f = \frac{1}{2} f_0 + f_1(\mu) \qquad f_0 \rightarrow \text{isotropic part}$$

$$\langle f \rangle = \int_{-1}^{1} f \, d\mu = f_0$$

$$\int_{-1}^{1} f_1 \, d\mu = 0$$

substitute in the p.a. diff. eq.

$$\frac{1}{2}\frac{\partial f_0}{\partial t} + \frac{\partial f_1}{\partial t} + \frac{1}{2} v\mu \frac{\partial f_0}{\partial z} + v\mu \frac{\partial f_1}{\partial z} = \frac{\partial}{\partial \mu}\left(D_{\mu\mu} \frac{\partial f_1}{\partial \mu}\right) \qquad (1)$$

Integrate over $\mu$ from $-1$ to $+1$

$$\frac{\partial f_0}{\partial t} + v\int_{-1}^{1} \mu \frac{\partial f_1}{\partial z} d\mu = D_{\mu\mu} \frac{\partial f_1}{\partial \mu}\Big]_{-1}^{+1} = 0$$

$$\boxed{\frac{\partial f_0}{\partial t} = -v\int_{-1}^{1} \mu \frac{\partial f_1}{\partial z} d\mu} \qquad (2)$$

multiply (2) x 1/2, subtract from (1)

$$\overset{1}{\frac{\partial f_1}{\partial t}} + \overset{2}{\frac{1}{2} v\mu \frac{\partial f_0}{\partial z}} + \overset{3}{v\mu \frac{\partial f_1}{\partial z}} = \overset{4}{\frac{\partial}{\partial \mu}\left(D_{\mu\mu}\frac{\partial f_1}{\partial \mu}\right)} + \overset{5}{v\int_{-1}^{1} \mu \frac{\partial f_1}{\partial z} d\mu}$$

because $T$, characteristic time scale & $L$ characteristic length scale are <u>large</u> compared to scattering time, $\tau$, and mean-free path $\lambda$

$$T \gg \tau$$
$$L \gg \lambda$$

and $f_1 \ll f_0$

only terms 2 & 4 are important in this eq.

$$\therefore \quad \frac{1}{2} v\mu \frac{\partial f_0}{\partial z} \simeq \frac{\partial}{\partial \mu}\left(D_{\mu\mu}\frac{\partial f_1}{\partial \mu}\right)$$

$$\Rightarrow \int_{-1}^{\mu} \frac{1}{2} v\mu' \frac{\partial f_0}{\partial z} d\mu' = D_{\mu\mu}\frac{\partial f_1}{\partial \mu}$$

$$\Rightarrow \frac{v}{4}(\mu^2 - 1)\frac{\partial f_0}{\partial z} = D_{\mu\mu}\frac{\partial f_1}{\partial \mu}$$

$$\Rightarrow \quad \frac{\partial f_1}{\partial \mu} = \frac{v}{4}(\mu^2-1)\frac{1}{D_{\mu\mu}}\frac{\partial f_0}{\partial z}$$

$$f_1 = \int_{-1}^{\mu} \frac{v}{4}(\mu'^2-1)\frac{1}{D_{\mu\mu}(\mu')}\frac{\partial f_0}{\partial z}\,\mu'$$

Insert this into (2) (boxed eq.)

$$\frac{\partial f_0}{\partial z} = \frac{v^2}{4}\int_{-1}^{1}\mu\, d\mu \frac{\partial}{\partial z}\left[\int_{-1}^{\mu}\frac{1-\mu'^2}{D_{\mu\mu}}\frac{\partial f_0}{\partial z}\,d\mu'\right]$$

$$= \frac{\partial}{\partial z}\left(\frac{v^2}{4}\int_{-1}^{1}\mu\, d\mu \int_{-1}^{\mu}\frac{1-\mu'^2}{D_{\mu\mu}}\,d\mu' \;\frac{\partial f_0}{\partial z}\right)$$

$$= \frac{\partial}{\partial z}\,\kappa\,\frac{\partial f_0}{\partial z}$$

where

$$\kappa = \frac{v^2}{4}\int_{-1}^{1}\mu\, d\mu \int_{-1}^{\mu}\frac{1-\mu'^2}{D_{\mu\mu}(\mu')}\,d\mu'$$

Change order of integration to give

$$K = \frac{v^2}{4}\int_{-1}^{1}\frac{1-\mu'^2}{D_{\mu\mu}(\mu')}\,d\mu'\int_{\mu}^{1}\mu\,d\mu$$

$\vdots$

$$\boxed{K = \frac{v^2}{4}\int_0^1 \frac{(1-\mu^2)^2\,d\mu}{D_{\mu\mu}}}$$

Earl, 1974
Lohmann, 1976

recall

$$\boxed{D_{\mu\mu} = \frac{\pi}{4}\Omega_0(1-\mu^2)\frac{k_r P(k_r)}{B_0^2}}$$

quasi-linear theory
Jokipii, 1966

$$\boxed{k_r = \frac{\Omega}{v\mu}}$$

resonant wave #

$P \rightarrow$ power spectrum

$K = \frac{1}{3}\lambda v \leftarrow$ particle speed

$\lambda \leftarrow$ mean free path

# Particle Transport: Energy change

recall eq. of motion of a charged particle

$$\frac{d\vec{p}}{dt} = q\underset{\sim}{E} + \frac{q}{c}\underset{\sim}{w} \times \underset{\sim}{B}$$

$$\underset{\sim}{w} \cdot \frac{d\underset{\sim}{p}}{dt} = q\,\underset{\sim}{w} \cdot \underset{\sim}{E}$$

$$\frac{dT}{dt} = q\,\underset{\sim}{w} \cdot \underset{\sim}{E} \qquad T = \text{kinetic energy}$$

$$\underset{\sim}{E} = -\frac{1}{c}\underset{\sim}{U} \times \underset{\sim}{B} \rightarrow \text{ideal MHD}$$

$$\Rightarrow \quad \frac{dT}{dt} = -\frac{q}{c}\underset{\sim}{w} \cdot (\underset{\sim}{U} \times \underset{\sim}{B})$$

T is in the lab frame – inertial frame.
Consider instead the frame moving w/ fluid

$$T' = T - \underset{\sim}{p} \cdot \underset{\sim}{U} + \frac{1}{2} m U^2$$

$$\Rightarrow \frac{dT'}{dt} = \frac{dT}{dt} - \frac{d}{dt}(\underset{\sim}{p} \cdot \underset{\sim}{U}) + \frac{d}{dt}\left(\frac{1}{2} m U^2\right)$$

$$\frac{dT'}{dt} = -\frac{q}{c}\underset{\sim}{w}\cdot(\underset{\sim}{U}\times\underset{\sim}{B}) - \underset{\sim}{p}\cdot\frac{d\underset{\sim}{U}}{dt} - \underset{\sim}{U}\cdot\underbrace{\frac{d\underset{\sim}{p}}{dt}}_{q\underset{\sim}{E}+\frac{q}{c}\underset{\sim}{w}\times\underset{\sim}{B}} + m\underset{\sim}{U}\cdot\frac{d\underset{\sim}{U}}{dt}$$

and $q\underset{\sim}{E}\cdot\underset{\sim}{U} = -\frac{q}{c}\underset{\sim}{U}\times\underset{\sim}{B}\cdot\underset{\sim}{U} = 0$

$$= -\frac{q}{c}\underset{\sim}{w}\cdot(\underset{\sim}{U}\times\underset{\sim}{B}) - \underset{\sim}{p}\cdot\frac{d\underset{\sim}{U}}{dt} - \frac{q}{c}\underset{\sim}{U}\cdot(\underset{\sim}{w}\times\underset{\sim}{B}) + m\underset{\sim}{U}\cdot\frac{d\underset{\sim}{U}}{dt}$$

these cancel after implementing a vector identity

$$= -(\underset{\sim}{p} - m\underset{\sim}{U})\cdot\frac{d\underset{\sim}{U}}{dt}$$

$$\frac{dT'}{dt} = -\underset{\sim}{p}'\cdot\frac{d\underset{\sim}{U}}{dt}$$

$\frac{d}{dt} \rightarrow$ along particle trajectory

$$= -\underset{\sim}{p}'\cdot\left(\frac{\partial\underset{\sim}{U}}{\partial t} + \underset{\sim}{w}\cdot\nabla\underset{\sim}{U}\right) \longleftarrow$$ use convective derivative

Consider a collection of particles, distributed isotropically, then

$$\frac{d\langle T'\rangle}{dt} = \left\langle -\underset{\sim}{p}' \cdot \left(\frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{w}' \cdot \nabla \underset{\sim}{u}\right)\right\rangle$$

avg. over isotropic dist.

$$= -\langle \underset{\sim}{p}' \cdot (\underset{\sim}{w}' \cdot \nabla \underset{\sim}{u})\rangle$$

assume $\underset{\sim}{u}$ varies slowly in time compared to scattering time

index not.

$$= -\langle p_i' w_j' \partial_j u_i \rangle$$

$$= -\underbrace{\langle p_i' w_j' \rangle}_{\frac{1}{3} p' w' \delta_{ij}} \partial_j u_i$$

assume u varies in space slowly compared to $\lambda$ (mean free path)

kronecker delta

$$= -\frac{1}{3} p' w' \nabla \cdot \underset{\sim}{u}$$

recall $dT'/dt = w' dp'/dt$

$$\Rightarrow \boxed{\frac{d\langle p\rangle}{dt} = -\frac{1}{3} p' \nabla \cdot \underset{\sim}{u}}$$

energy change of an isotropic dist. of particles

# Cosmic-Ray Transport Equation

$$\frac{\partial f}{\partial t} + \underset{\sim}{U}\cdot\nabla f = \nabla\cdot(\underset{\approx}{K}\cdot\nabla f) + \frac{1}{3}\nabla\cdot\underset{\sim}{U}\frac{\partial f}{\partial \ln p} + S - L$$

advection w/ fluid

spatial diffusion <u>and</u> drifts

energy change

sources losses

where,

$$K_{ij} = K_\perp \delta_{ij} + (K_\parallel - K_\perp)\frac{B_i B_j}{B^2} + \epsilon_{ijk} K_A \frac{B_k}{B}$$

$K_\perp$ = cross-field diffusion coeff.

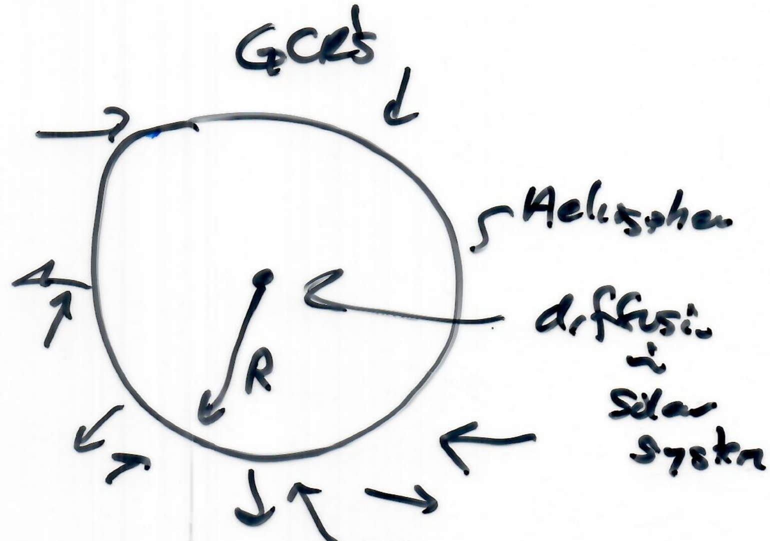
$K_\parallel$ = parallel " " (previous discussion)

$K_A$ = anti-symmetric component

→ this is where drifts are!

## GCR modulation

Consider the spherical coord. rep. of the CR transport eq. (Parker equation) in steady state




$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(\kappa r^2 \frac{\partial f}{\partial r}\right) - U\frac{\partial f}{\partial r} - \frac{2Up}{3r}\frac{\partial f}{\partial p} = 0$$

this can be written

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\left(\kappa \frac{\partial f}{\partial r} - Uf\right)\right] + \frac{2U}{3rp^2}\frac{\partial}{\partial p}(p^3 f) = 0$$

if one ignores the last term, we find

$$\boxed{f(r,p) = f(R,p)\, e^{-\int_r^R \frac{U\, dr'}{\kappa(r',p)}}}$$

commonly used "model" (not a good one, tho)