How Wave-Particle Interactions Shape Particle Distribution Functions in the Solar Wind

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Outline

I. Resonant Wave-Particle Interactions and the Lost Art of Quasilinear Theory

  A. ion cyclotron heating

  B. self-induced scattering of the electron *strahl* by oblique whistler waves

  C. (cosmic-ray streaming instability, deceleration of alpha-particle beams, limits on proton temperature anisotropy)

II. Dissipation of Solar-Wind Turbulence by Non-Resonant Stochastic Ion Heating

Collaborators

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- Florida Institute of Technology: J. Perez
- Oxford: A. Schekochihin
- Princeton: L. Arzamasskiy, M. Kunz
- U. Arizona: K. Klein
- UC Berkeley: S. Bale, A. Mallet, , M. Pulupa, E. Quataert, C. Salem
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- U. Michigan: J. Kasper, K. Klein, B. van der Holst
- UNH: D. Borovikov, S. Chernoglazov, K. Germaschewski, J. Hollweg, P. Isenberg, I. Hoppock, M. Lee, S. Markovskii, B. Vasquez
- Special acknowledgement: R. Kulsrud

Quasilinear Theory

(Yakimenko 1963; Kennel & Engelmann 1966; Stix 1992)

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla f + \frac{q}{m} \left(\boldsymbol{E} + \frac{1}{c} \boldsymbol{v} \times \boldsymbol{B} \right) \cdot \nabla_v f = 0 \tag{1}$$

Vlasov equation

Quasilinear Theory

(Yakimenko 1963; Kennel & Engelmann 1966; Stix 1992)

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla f + \frac{q}{m}\left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v} \times \boldsymbol{B}\right) \cdot \nabla_v f = 0 \tag{1}$$

We want to use the Vlasov equation to figure out how the distribution function evolves in the presence of plasma waves.

We could just solve (1) on a computer.

But aren't there any organizing principles we could use to understand wave-particle interactions? Is there a conceptual framework we could use to reason out how the distribution function evolves over time in the presence of plasma waves?

Let's look for such a conceptual framework by solving (1) using a perturbative technique.

Quasilinear Theory

(Yakimenko 1963; Kennel & Engelmann 1966; Stix 1992)

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$$f = f_0(\boldsymbol{x}, \boldsymbol{v}, t) + f_1(\boldsymbol{x}, \boldsymbol{v}, t) \qquad \boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{B}_1(\boldsymbol{x}, t) \qquad \boldsymbol{E} = \boldsymbol{E}_1(\boldsymbol{x}, t)$$

B_0 is a uniform background magnetic field.

f_0 is the background or equilibrium plasma distribution function

E_1 and B_1 represent a collection of waves, which could be slowly growing or slowly decaying. We're going to treat E_1 and B_1 as known.

f_1 represents the response of the plasma to these waves

Our goal is to find how f_0 varies over times much longer than the wave periods.

Quasilinear Theory

(Yakimenko 1963; Kennel & Engelmann 1966; Stix 1992)

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla f + \frac{q}{m}\left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v} \times \boldsymbol{B}\right) \cdot \nabla_v f = 0 \tag{1}$$

$$f = f_0(\boldsymbol{x}, \boldsymbol{v}, t) + f_1(\boldsymbol{x}, \boldsymbol{v}, t) \qquad \boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{B}_1(\boldsymbol{x}, t) \qquad \boldsymbol{E} = \boldsymbol{E}_1(\boldsymbol{x}, t)$$

Let's plug these expressions into (1), and then separately equate all the "zeroth-order terms" and all the "first-order terms".

Quasilinear Theory

(Yakimenko 1963; Kennel & Engelmann 1966; Stix 1992)

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla f + \frac{q}{m} \left(\boldsymbol{E} + \frac{1}{c} \boldsymbol{v} \times \boldsymbol{B} \right) \cdot \nabla_v f = 0 \tag{1}$$

$$f = f_0(\boldsymbol{x}, \boldsymbol{v}, t) + f_1(\boldsymbol{x}, \boldsymbol{v}, t) \qquad \boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{B}_1(\boldsymbol{x}, t) \qquad \boldsymbol{E} = \boldsymbol{E}_1(\boldsymbol{x}, t)$$

$$\frac{\partial f_0}{\partial t} +$$

Quasilinear Theory

(Yakimenko 1963; Kennel & Engelmann 1966; Stix 1992)

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla f + \frac{q}{m} \left(\boldsymbol{E} + \frac{1}{c} \boldsymbol{v} \times \boldsymbol{B} \right) \cdot \nabla_v f = 0 \tag{1}$$

$$f = f_0(\boldsymbol{x}, \boldsymbol{v}, t) + f_1(\boldsymbol{x}, \boldsymbol{v}, t) \qquad \boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{B}_1(\boldsymbol{x}, t) \qquad \boldsymbol{E} = \boldsymbol{E}_1(\boldsymbol{x}, t)$$

$$\frac{\partial f_0}{\partial t} + \boldsymbol{v} \cdot \nabla f_0 +$$

Quasilinear Theory

(Yakimenko 1963; Kennel & Engelmann 1966; Stix 1992)

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla f + \frac{q}{m}\left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v} \times \boldsymbol{B}\right) \cdot \nabla_v f = 0 \tag{1}$$

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$$\frac{\partial f_0}{\partial t} + \boldsymbol{v} \cdot \nabla f_0 + \frac{q}{mc}(\boldsymbol{v} \times \boldsymbol{B}_0) \cdot \nabla_v f_0 = 0$$

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$$\frac{\partial f_0}{\partial t} + \boldsymbol{v} \cdot \nabla f_0 + \frac{q}{mc}(\boldsymbol{v} \times \boldsymbol{B}_0) \cdot \nabla_v f_0 = 0 \rightarrow f_0(\boldsymbol{x}, \boldsymbol{v}, t) = f_0(v_\perp, v_\parallel)$$

Here, we are using cylindrical coordinates $(v_\perp, v_\parallel, \theta)$ in velocity space, where the cylindrical axis is aligned with $\boldsymbol{B}_0$. Soon, we will set $\boldsymbol{B}_0 \rightarrow B_0 \hat{\boldsymbol{z}}$, and $v_\parallel$ will become v_z.

(technically, f_0 varies in time over time scales much longer than the wave periods. But here the variable t describes time variations over times comparable to the wave periods, and f_0 doesn't vary on this “fast” time scale.

Quasilinear Theory

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$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla f + \frac{q}{m}\left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v} \times \boldsymbol{B}\right) \cdot \nabla_v f = 0 \tag{1}$$

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Now let's collect all the “first-order” terms in (1).

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$$f = f_0(\boldsymbol{x}, \boldsymbol{v}, t) + f_1(\boldsymbol{x}, \boldsymbol{v}, t) \qquad \boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{B}_1(\boldsymbol{x}, t) \qquad \boldsymbol{E} = \boldsymbol{E}_1(\boldsymbol{x}, t)$$

$$\frac{\partial f_0}{\partial t} + \boldsymbol{v} \cdot \nabla f_0 + \frac{q}{mc} (\boldsymbol{v} \times \boldsymbol{B}_0) \cdot \nabla_v f_0 = 0 \rightarrow f_0(\boldsymbol{x}, \boldsymbol{v}, t) = f_0(v_\perp, v_\parallel)$$

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(Yakimenko 1963; Kennel & Engelmann 1966; Stix 1992)

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla f + \frac{q}{m}\left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v} \times \boldsymbol{B}\right) \cdot \nabla_v f = 0 \tag{1}$$

$$f = f_0(\boldsymbol{x}, \boldsymbol{v}, t) + f_1(\boldsymbol{x}, \boldsymbol{v}, t) \qquad \boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{B}_1(\boldsymbol{x}, t) \qquad \boldsymbol{E} = \boldsymbol{E}_1(\boldsymbol{x}, t)$$

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$$\frac{\partial f_1}{\partial t} + \boldsymbol{v} \cdot \nabla f_1 +$$

Quasilinear Theory

(Yakimenko 1963; Kennel & Engelmann 1966; Stix 1992)

$$\frac{\partial f}{\partial t} + \boldsymbol{v}\cdot\nabla f + \frac{q}{m}\left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v}\times\boldsymbol{B}\right)\cdot\nabla_v f = 0 \tag{1}$$

$$f = f_0(\boldsymbol{x},\boldsymbol{v},t) + f_1(\boldsymbol{x},\boldsymbol{v},t) \qquad \boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{B}_1(\boldsymbol{x},t) \qquad \boldsymbol{E} = \boldsymbol{E}_1(\boldsymbol{x},t)$$

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$$\frac{\partial f_1}{\partial t} + \boldsymbol{v}\cdot\nabla f_1 + \frac{q}{mc}(\boldsymbol{v}\times\boldsymbol{B}_0)\cdot\nabla_v f_1 = -\frac{q}{m}\left(\boldsymbol{E}_1 + \frac{1}{c}\boldsymbol{v}\times\boldsymbol{B}_1\right)\cdot\nabla_v f_0$$

Quasilinear Theory

(Yakimenko 1963; Kennel & Engelmann 1966; Stix 1992)

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla f + \frac{q}{m}\left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v} \times \boldsymbol{B}\right) \cdot \nabla_v f = 0 \tag{1}$$

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Difficult-looking equation. How do we solve this equation for $f_1(\boldsymbol{x},\boldsymbol{v},t)$ if we know $\boldsymbol{E}_1$, $\boldsymbol{B}_1$, $\boldsymbol{B}_0$, and f_0?

Quasilinear Theory

(Yakimenko 1963; Kennel & Engelmann 1966; Stix 1992)

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla f + \frac{q}{m} \left(\boldsymbol{E} + \frac{1}{c} \boldsymbol{v} \times \boldsymbol{B} \right) \cdot \nabla_v f = 0 \qquad (1)$$

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Difficult-looking equation. How do we solve this equation for $f_1(\boldsymbol{x}, \boldsymbol{v}, t)$ if we know $\boldsymbol{E}_1$, $\boldsymbol{B}_1$, $\boldsymbol{B}_0$, and f_0?

Method of characteristics!

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$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla f + \frac{q}{m}\left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v} \times \boldsymbol{B}\right) \cdot \nabla_v f = 0 \tag{1}$$

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$$\frac{\partial f_0}{\partial t} + \boldsymbol{v} \cdot \nabla f_0 + \frac{q}{mc}(\boldsymbol{v} \times \boldsymbol{B}_0) \cdot \nabla_v f_0 = 0 \to f_0(\boldsymbol{x}, \boldsymbol{v}, t) = f_0(v_\perp, v_\parallel)$$

$$\frac{\partial f_1}{\partial t} + \boldsymbol{v} \cdot \nabla f_1 + \frac{q}{mc}(\boldsymbol{v} \times \boldsymbol{B}_0) \cdot \nabla_v f_1 = -\frac{q}{m}\left(\boldsymbol{E}_1 + \frac{1}{c}\boldsymbol{v} \times \boldsymbol{B}_1\right) \cdot \nabla_v f_0$$

Let $f_1(\boldsymbol{x}, \boldsymbol{v}, t) = f_1(\boldsymbol{x}(t), \boldsymbol{v}(t), t)$, where $d\boldsymbol{x}/dt = \boldsymbol{v}$, $d\boldsymbol{v}/dt = (q/mc)(\boldsymbol{v} \times \boldsymbol{B}_0)$:

Quasilinear Theory

(Yakimenko 1963; Kennel & Engelmann 1966; Stix 1992)

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla f + \frac{q}{m} \left(\boldsymbol{E} + \frac{1}{c} \boldsymbol{v} \times \boldsymbol{B} \right) \cdot \nabla_v f = 0 \tag{1}$$

$$f = f_0(\boldsymbol{x}, \boldsymbol{v}, t) + f_1(\boldsymbol{x}, \boldsymbol{v}, t) \qquad \boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{B}_1(\boldsymbol{x}, t) \qquad \boldsymbol{E} = \boldsymbol{E}_1(\boldsymbol{x}, t)$$

$$\frac{\partial f_0}{\partial t} + \boldsymbol{v} \cdot \nabla f_0 + \frac{q}{mc} (\boldsymbol{v} \times \boldsymbol{B}_0) \cdot \nabla_v f_0 = 0 \rightarrow f_0(\boldsymbol{x}, \boldsymbol{v}, t) = f_0(v_\perp, v_\parallel)$$

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Let $f_1(\boldsymbol{x}, \boldsymbol{v}, t) = f_1(\boldsymbol{x}(t), \boldsymbol{v}(t), t)$, where $d\boldsymbol{x}/dt = \boldsymbol{v}$, $d\boldsymbol{v}/dt = (q/mc)(\boldsymbol{v} \times \boldsymbol{B}_0)$:

$$\frac{\mathrm{d} f_1}{\mathrm{d} t}$$

Quasilinear Theory

(Yakimenko 1963; Kennel & Engelmann 1966; Stix 1992)

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla f + \frac{q}{m}\left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v} \times \boldsymbol{B}\right) \cdot \nabla_v f = 0 \tag{1}$$

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$$\frac{\partial f_1}{\partial t} + \boldsymbol{v} \cdot \nabla f_1 + \frac{q}{mc}(\boldsymbol{v} \times \boldsymbol{B}_0) \cdot \nabla_v f_1 = -\frac{q}{m}\left(\boldsymbol{E}_1 + \frac{1}{c}\boldsymbol{v} \times \boldsymbol{B}_1\right) \cdot \nabla_v f_0$$

Let $f_1(\boldsymbol{x}, \boldsymbol{v}, t) = f_1(\boldsymbol{x}(t), \boldsymbol{v}(t), t)$, where $d\boldsymbol{x}/dt = \boldsymbol{v}$, $d\boldsymbol{v}/dt = (q/mc)(\boldsymbol{v} \times \boldsymbol{B}_0)$:

$$\frac{\mathrm{d}f_1}{\mathrm{d}t} = -\frac{q}{m}\left(\boldsymbol{E}_1(\boldsymbol{x}(t), t) + \frac{1}{c}\boldsymbol{v}(t) \times \boldsymbol{B}_1(\boldsymbol{x}(t), t)\right) \cdot \nabla_v f_0 \tag{2}$$

Quasilinear Theory

(Yakimenko 1963; Kennel & Engelmann 1966; Stix 1992)

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla f + \frac{q}{m} \left(\boldsymbol{E} + \frac{1}{c} \boldsymbol{v} \times \boldsymbol{B} \right) \cdot \nabla_v f = 0 \tag{1}$$

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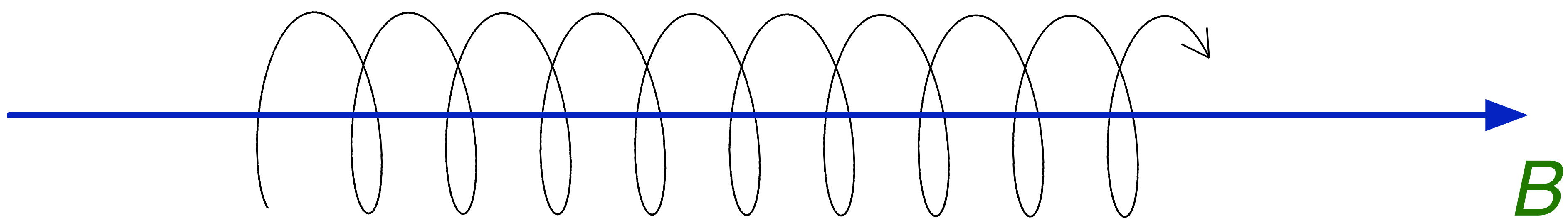
Let $f_1(\boldsymbol{x}, \boldsymbol{v}, t) = f_1(\boldsymbol{x}(t), \boldsymbol{v}(t), t)$, where $d\boldsymbol{x}/dt = \boldsymbol{v}$, $d\boldsymbol{v}/dt = (q/mc)(\boldsymbol{v} \times \boldsymbol{B}_0)$:

$$\frac{\mathrm{d} f_1}{\mathrm{d} t} = -\frac{q}{m} \left(\boldsymbol{E}_1(\boldsymbol{x}(t), t) + \frac{1}{c} \boldsymbol{v}(t) \times \boldsymbol{B}_1(\boldsymbol{x}(t), t) \right) \cdot \nabla_v f_0 \tag{2}$$

Solve for $\boldsymbol{x}(t)$ and $\boldsymbol{v}(t)$; integrate (2) to find f_1; plug f_1 into 3$^{\text{rd}}$ term in (1); and average.

Single Particle Motion in a Uniform Magnetic FIeld



$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{v}$$

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \frac{q}{mc}(\boldsymbol{v} \times \boldsymbol{B}_0)$$

$$\boldsymbol{B}_0 = B_0\hat{\boldsymbol{z}} \longrightarrow v_z = \text{ constant} \qquad v_\perp = \sqrt{v_x^2 + v_y^2} = \text{ constant}$$

Helical motion. Cyclotron frequency $\Omega = \dfrac{qB_0}{mc}$, gyroradius $= \rho = v_\perp/\Omega$.

Quasilinear Theory

(Yakimenko 1963; Kennel & Engelmann 1966; Stix 1992)

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla f + \frac{q}{m} \left(\boldsymbol{E} + \frac{1}{c} \boldsymbol{v} \times \boldsymbol{B} \right) \cdot \nabla_v f = 0 \tag{1}$$

$$f = f_0(\boldsymbol{x}, \boldsymbol{v}, t) + f_1(\boldsymbol{x}, \boldsymbol{v}, t) \qquad \boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{B}_1(\boldsymbol{x}, t) \qquad \boldsymbol{E} = \boldsymbol{E}_1(\boldsymbol{x}, t)$$

$$\frac{\partial f_0}{\partial t} + \boldsymbol{v} \cdot \nabla f_0 + \frac{q}{mc} (\boldsymbol{v} \times \boldsymbol{B}_0) \cdot \nabla_v f_0 = 0 \rightarrow f_0(\boldsymbol{x}, \boldsymbol{v}, t) = f_0(v_\perp, v_\parallel)$$

$$\frac{\partial f_1}{\partial t} + \boldsymbol{v} \cdot \nabla f_1 + \frac{q}{mc} (\boldsymbol{v} \times \boldsymbol{B}_0) \cdot \nabla_v f_1 = -\frac{q}{m} \left(\boldsymbol{E}_1 + \frac{1}{c} \boldsymbol{v} \times \boldsymbol{B}_1 \right) \cdot \nabla_v f_0$$

Let $f_1(\boldsymbol{x}, \boldsymbol{v}, t) = f_1(\boldsymbol{x}(t), \boldsymbol{v}(t), t)$, where $d\boldsymbol{x}/dt = \boldsymbol{v}$, $d\boldsymbol{v}/dt = (q/mc)(\boldsymbol{v} \times \boldsymbol{B}_0)$:

$$\frac{\mathrm{d} f_1}{\mathrm{d} t} = -\frac{q}{m} \left(\boldsymbol{E}_1(\boldsymbol{x}(t), t) + \frac{1}{c} \boldsymbol{v}(t) \times \boldsymbol{B}_1(\boldsymbol{x}(t), t) \right) \cdot \nabla_v f_0 \tag{2}$$

Solve for $\boldsymbol{x}(t)$ and $\boldsymbol{v}(t)$; integrate (2) to find f_1; plug f_1 into 3^{rd} term in (1); and average. Then, after *considerable* algebra, you find:

Quasilinear Theory

(Yakimenko 1963; Kennel & Engelmann 1966; Stix 1992)

(Henceforth, we follow convention in dropping the 0 subscript on f and the 1 subscript on the Es)

$$\frac{\partial f}{\partial t} = \lim_{V\to\infty} \sum_{n=-\infty}^{\infty} \frac{\pi q^2}{m^2} \int \frac{d^3k}{(2\pi)^3 V v_\perp} G\, v_\perp \delta(\omega_{kr} - k_\parallel v_\parallel - n\Omega) |\psi_{n,k}|^2 G\, f,$$

$$G \equiv \left(1 - \frac{k_\parallel v_\parallel}{\omega_{kr}}\right) \frac{\partial}{\partial v_\perp} + \frac{k_\parallel v_\perp}{\omega_{kr}} \frac{\partial}{\partial v_\parallel}$$

(J_n = Bessel function of first kind and degree n.)

$$\psi_{n,k} = \frac{1}{\sqrt{2}} \left[E_{k,r} e^{i\phi} J_{n+1}(\sigma) + E_{k,l} e^{-i\phi} J_{n-1}(\sigma)\right] + \frac{v_\parallel}{v_\perp} E_{kz} J_n(\sigma) \qquad \sigma = k_\perp v_\perp / \Omega$$

Quasilinear Theory

(Yakimenko 1963; Kennel & Engelmann 1966; Stix 1992)

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla f + \frac{q}{m} \left(\boldsymbol{E} + \frac{1}{c} \boldsymbol{v} \times \boldsymbol{B} \right) \cdot \nabla_v f = 0, \tag{1}$$

$$f = f_0(\boldsymbol{v}) + f_1(\boldsymbol{x}, \boldsymbol{v}, t) \qquad \boldsymbol{B} = \boldsymbol{B}_0 + \boldsymbol{B}_1 \qquad \boldsymbol{E} = \boldsymbol{E}_1$$

Solve for f_1 in terms of f_0, $\boldsymbol{E}_1$, $\boldsymbol{B}_1$; plug f_1 into 3rd term in (1); average:

$$\frac{\partial f}{\partial t} = \lim_{V \to \infty} \sum_{n=-\infty}^{\infty} \frac{\pi q^2}{m^2} \int \frac{d^3 k}{(2\pi)^3 V v_\perp} G\, v_\perp \delta(\omega_{kr} - k_\parallel v_\parallel - n\Omega) |\psi_{n,k}|^2 G\, f,$$

$$G \equiv \left(1 - \frac{k_\parallel v_\parallel}{\omega_{kr}} \right) \frac{\partial}{\partial v_\perp} + \frac{k_\parallel v_\perp}{\omega_{kr}} \frac{\partial}{\partial v_\parallel}$$

$$\psi_{n,k} = \frac{1}{\sqrt{2}} \left[E_{k,r} e^{i\phi} J_{n+1}(\sigma) + E_{k,l} e^{-i\phi} J_{n-1}(\sigma) \right] + \frac{v_\parallel}{v_\perp} E_{kz} J_n(\sigma) \qquad \sigma = k_\perp v_\perp / \Omega$$

Quasilinear Theory

(Yakimenko 1963; Kennel & Engelmann 1966; Stix 1992)

$$\frac{\partial f}{\partial t} + \boldsymbol{v}\cdot\nabla f + \frac{q}{m}\left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v}\times\boldsymbol{B}\right)\cdot\nabla_v f = 0, \tag{1}$$

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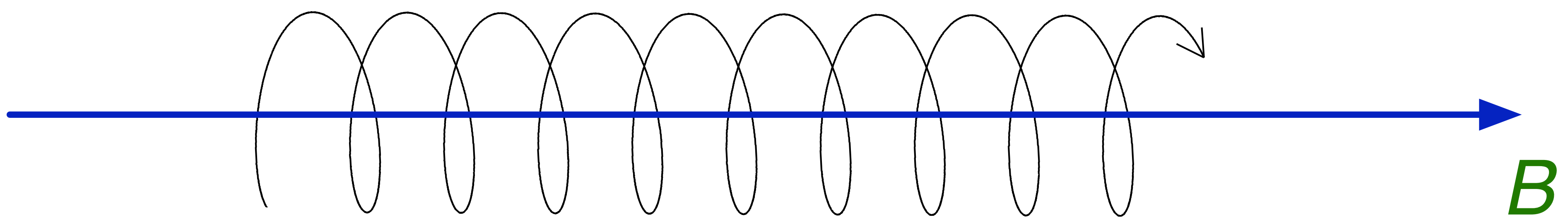
Wave-Particle Resonance Condition



- Consider $\delta\vec{E} = \delta\vec{E}_0 \cos(\vec{k}\cdot\vec{x} - \omega t)$

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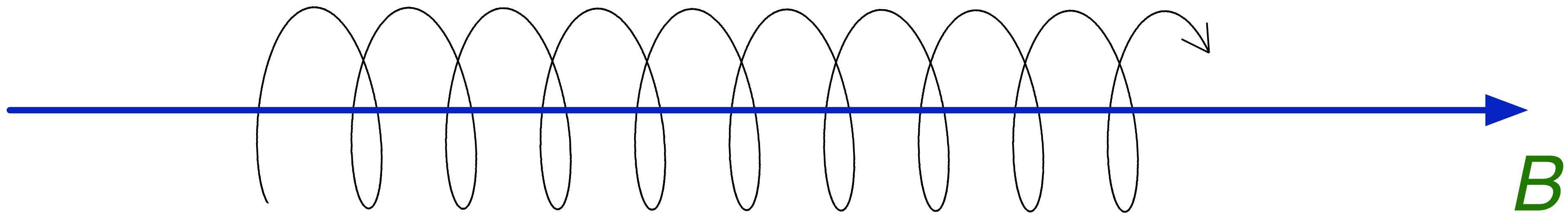

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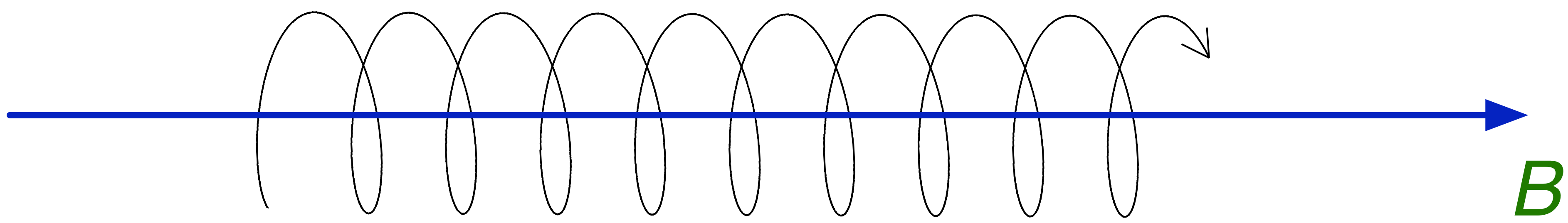

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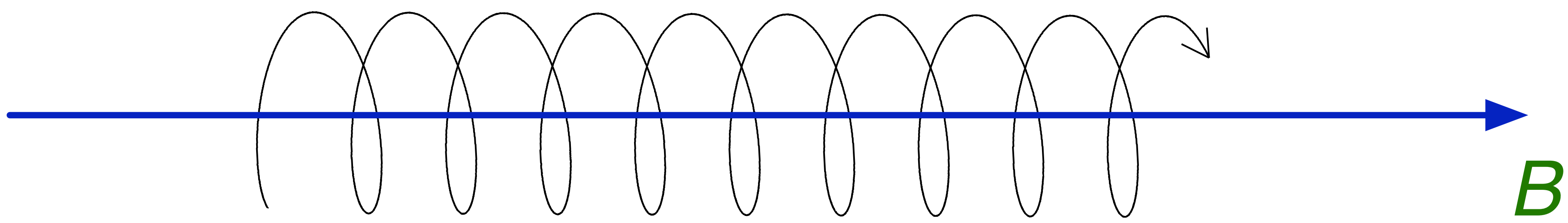

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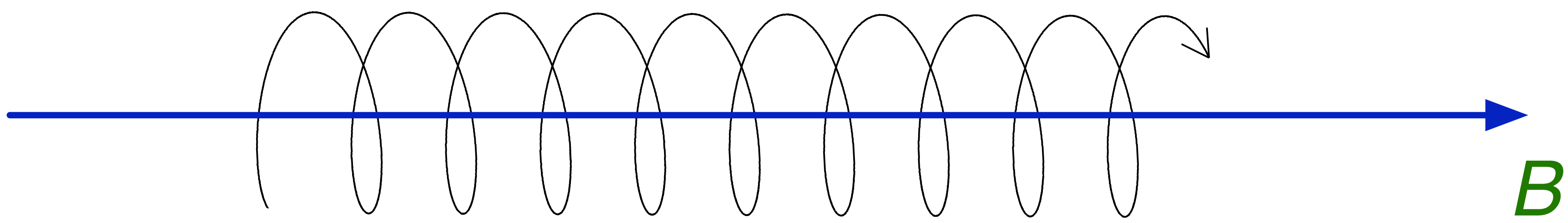

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- Wave-particle resonance when $\omega - k_{||}v_{||} = n\Omega$

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what kind of equation is this?

$$\frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial x^2}$$

Physically, what is the difference between a system described by the top equation, and a system described by the bottom equation?

$$\frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial x^2}$$

$$\frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial y^2}$$

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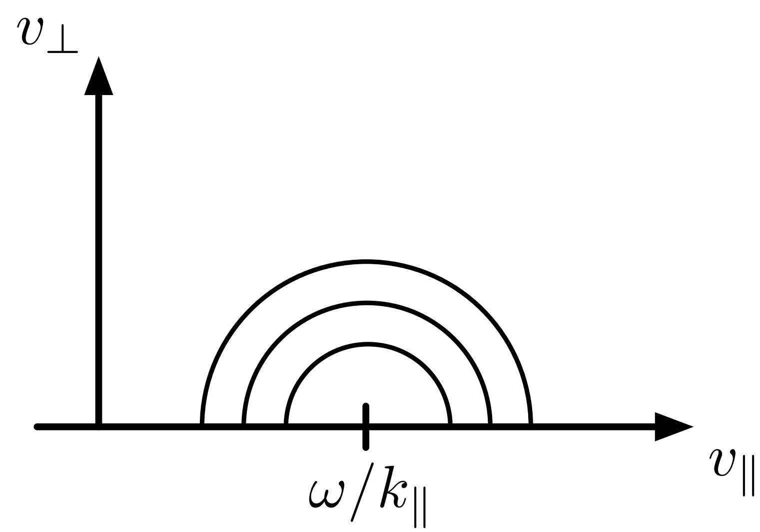
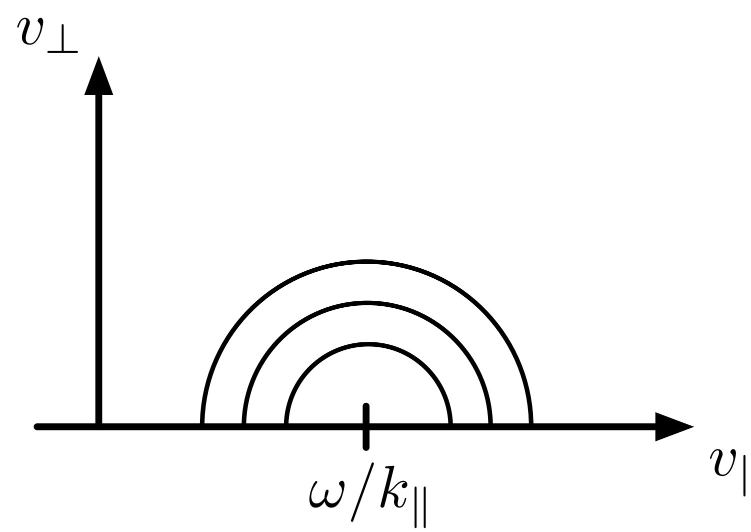


G is a derivative along a curve of constant energy K' in the "wave frame," which moves along $\boldsymbol{B}_0$ at velocity $\omega_{kr}/k_\|$, where

$$K' = \frac{m}{2}\left[v_\perp^2 + \left(v_\| - \frac{\omega_{kr}}{k_\|}\right)^2\right]$$

E.g., if $f(v_\perp, v_\|) = f(K')$, then

$$G(f) = \left(1 - \frac{k_\| v_\|}{\omega_{kr}}\right)\frac{\partial f}{\partial v_\perp} + \frac{k_\| v_\perp}{\omega_{kr}}\frac{\partial f}{\partial v_\|}$$



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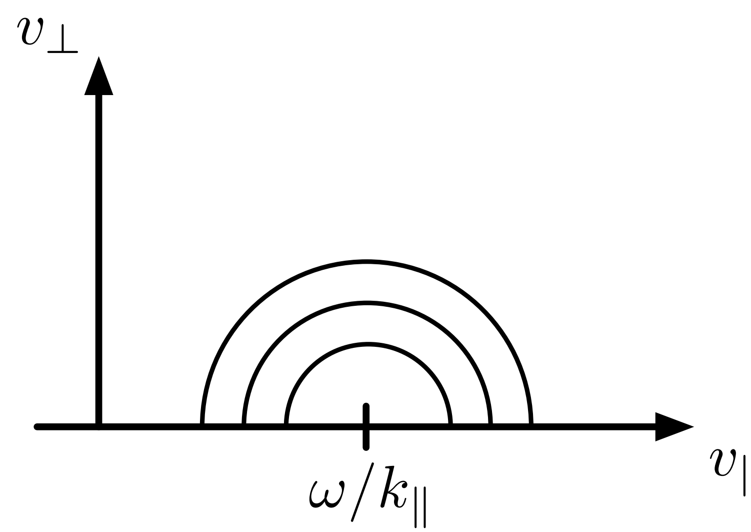

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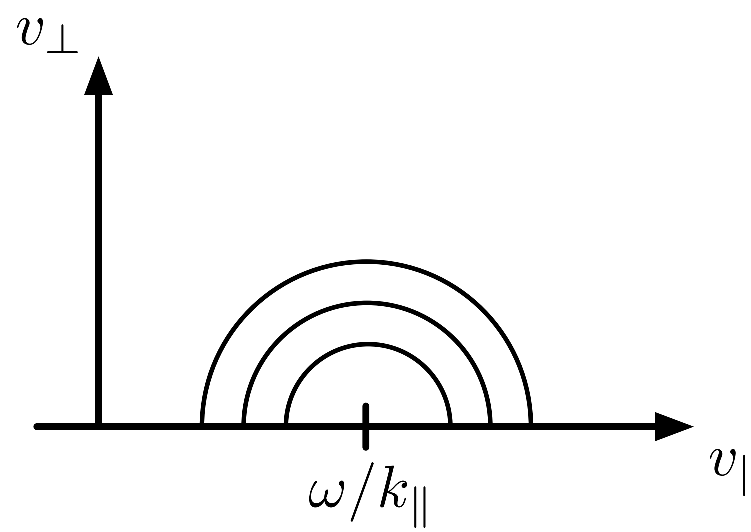

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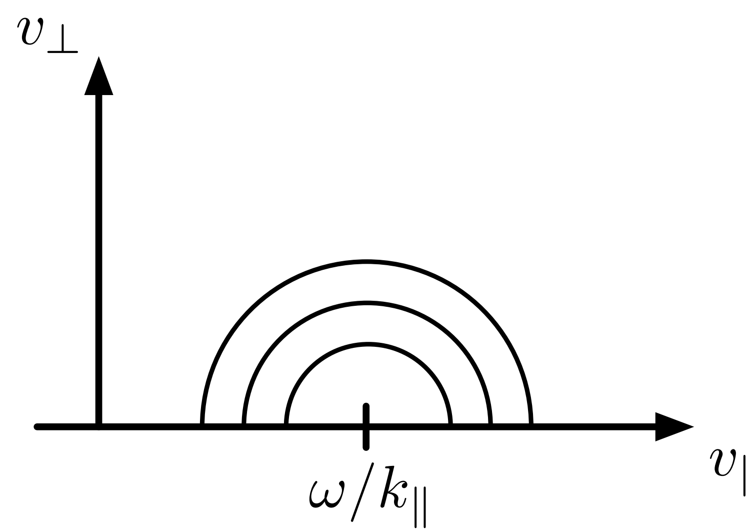

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So, particles diffuse in velocity space, along curves of constant energy in the wave frame! (But only if they satisfy resonance condition.)

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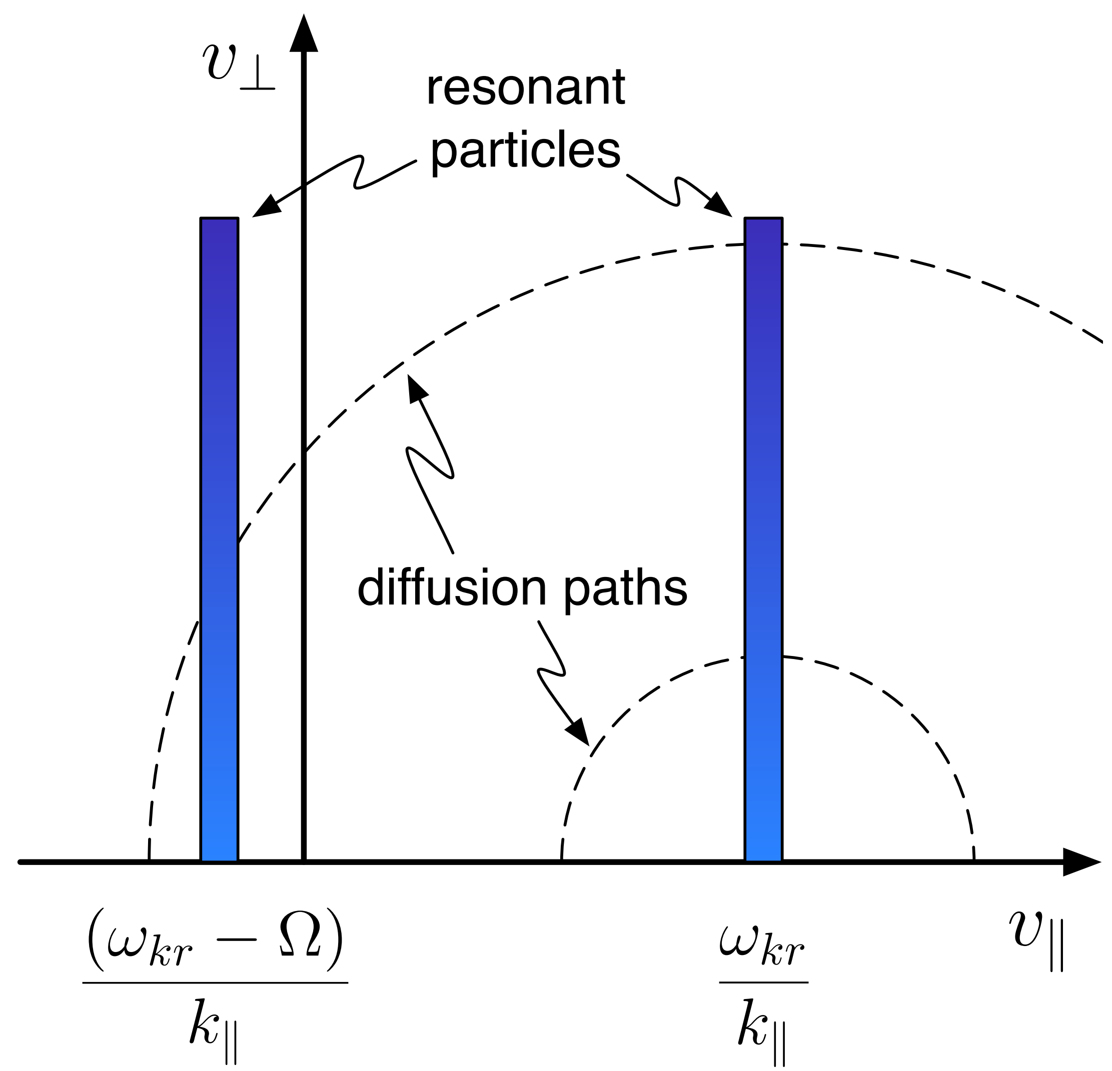
Note that the diffusion coefficient here is related to $|\psi_{n,k}|^2$ — i.e., the larger the wave amplitudes, the faster the particles diffuse in velocity space.

Energy conservation in wave frame

- Conisder a frame moving at $\vec{u} = \hat{b}\omega/k_{\|}$ with respect to the plasma
- the wave frequency in this frame is $\omega - k_{\|}u = 0$
- fluctuations are static: $\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0.$
- $\longrightarrow \vec{E} = -\nabla\Phi$
- Energy gain $= \Delta\mathcal{E} = q\Delta\Phi \longrightarrow$ energy gain can not accumulate over time
- Energy effectively conserved in this frame, but particle direction can change (pitch-angle scattering).

Types of Resonant Wave-Particle Interactions

- Wave-particle resonance condition: $\omega_{kr} - k_\parallel v_\parallel = n\Omega$.
- Landau damping (LD): $n = 0$, particles pushed by $\vec{E}$.
- Transit-time damping (TTD): $n = 0$, particles pushed by $\mu \nabla B$.
- Cyclotron damping (CD): $n \neq 0$.
- In the $v_\parallel - v_\perp$ plane, resonant particles diffuse along semi-circles centered on $v_\parallel = \omega_{kr}/k_\parallel$, because $\nabla \times \vec{E} = 0$ in the wave frame. The condition $E = -\nabla \Phi$ in the wave frame with Φ bounded means that there can be no secular energy gain in the wave frame.
- Hence, LD and TTD lead to parallel heating. CD can lead to perpendicular heating.



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energy conservation
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the importance of wave polarization

$$\boxed{\psi_{n,k} = \frac{1}{\sqrt{2}} \left[E_{k,r} e^{i\phi} J_{n+1}(\sigma) + E_{k,l} e^{-i\phi} J_{n-1}(\sigma) \right] + \frac{v_\parallel}{v_\perp} E_{kz} J_n(\sigma) \qquad \sigma = k_\perp v_\perp / \Omega}$$

Example 1: Ion Cyclotron Heating by Parallel-Propagating Alfvén/Ion-Cylotron Waves

(E.g., Hollweg & Isenberg 2002)

Ion Cyclotron Heating

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- When $\sigma \to 0$, $J_n(\sigma) \to 0$ unless $n = 0$. $(J_0(0) = 1.)$

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- Consider ions interacting with parallel-propagating (i.e., $k_\perp = 0$) Alfvén/ion-cyclotron waves with no parallel electric field ($E_{kz} = 0$).
- These waves are left circularly polarized, so $E_{k,r} = 0$
- Of all the $\psi_{n,k}$, only one is non-zero — which one?

Ion Cyclotron Heating

(E.g., Hollweg & Isenberg 2002)

$$\psi_{n,k} = \frac{1}{\sqrt{2}}\left[E_{k,r}e^{i\phi}J_{n+1}(\sigma) + E_{k,l}e^{-i\phi}J_{n-1}(\sigma)\right] + \frac{v_\parallel}{v_\perp}E_{kz}J_n(\sigma) \qquad \sigma = k_\perp v_\perp/\Omega$$

- When $\sigma \to 0$, $J_n(\sigma) \to 0$ unless $n = 0$. ($J_0(0) = 1$.)
- Consider ions interacting with parallel-propagating (i.e., $k_\perp = 0$) Alfvén/ion-cyclotron waves with no parallel electric field ($E_{kz} = 0$).
- These waves are left circularly polarized, so $E_{k,r} = 0$
- Of all the $\psi_{n,k}$, only one is non-zero — which one?
- the only nonzero $\psi_{n,k}$ is $\psi_{1,k}$.

Quasilinear Theory

(Yakimenko 1963; Kennel & Engelmann 1966; Stix 1992)

This means that for protons interacting with parallel-propagating Alfvén/ion-cyclotron waves, out of this infinite sum, we need only keep the n=1 term.

and for this term, the resonance condition is

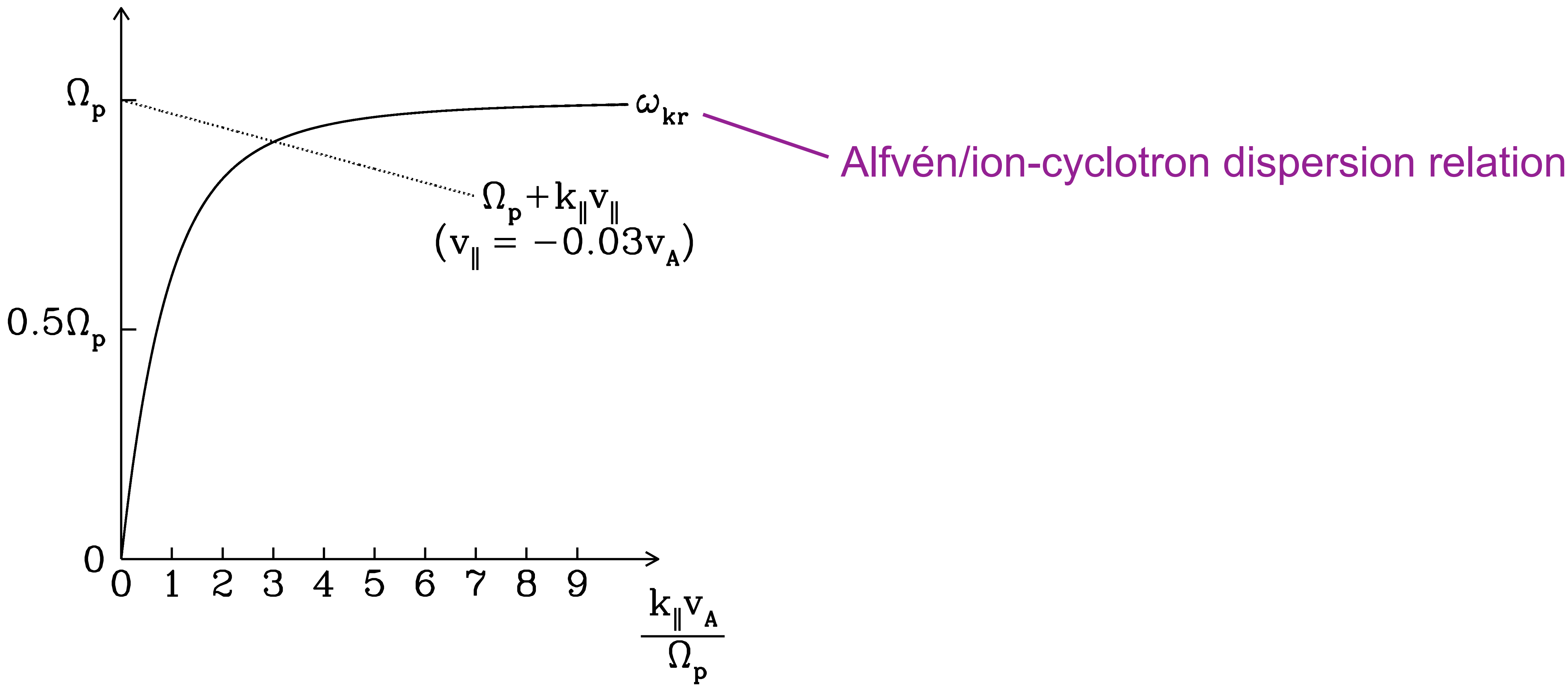
$$\omega_{kr} - k_\parallel v_\parallel = \Omega$$

$$\frac{\partial f}{\partial t} = \lim_{V\to\infty} \sum_{n=-\infty}^{\infty} \frac{\pi q^2}{m^2} \int \frac{d^3k}{(2\pi)^3 V v_\perp} G\, v_\perp \delta(\omega_{kr} - k_\parallel v_\parallel - n\Omega) |\psi_{n,k}|^2 G\, f,$$

$$G \equiv \left(1 - \frac{k_\parallel v_\parallel}{\omega_{kr}}\right) \frac{\partial}{\partial v_\perp} + \frac{k_\parallel v_\perp}{\omega_{kr}} \frac{\partial}{\partial v_\parallel}$$

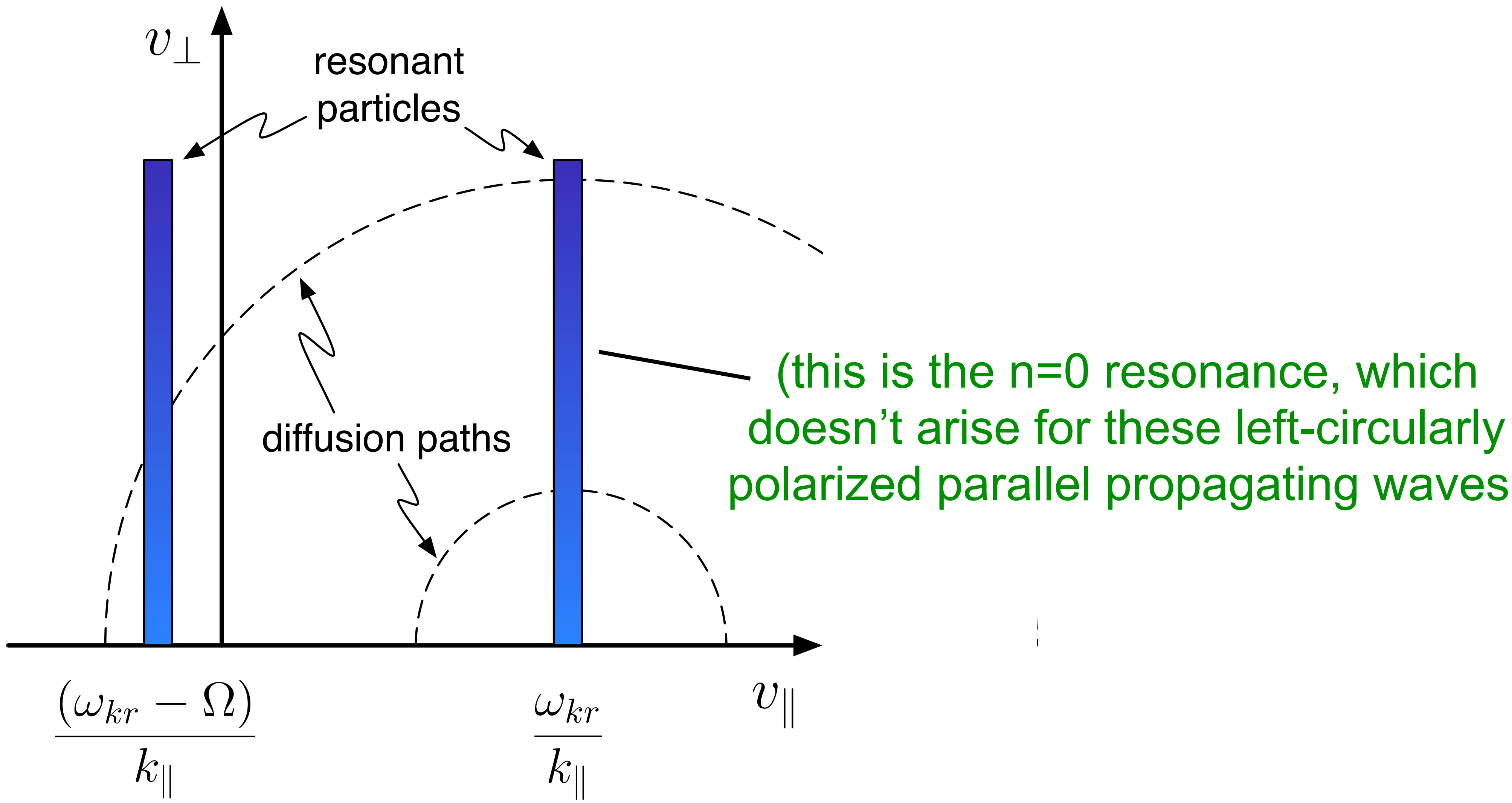
$$\psi_{n,k} = \frac{1}{\sqrt{2}} \left[E_{k,r} e^{i\phi} J_{n+1}(\sigma) + E_{k,l} e^{-i\phi} J_{n-1}(\sigma)\right] + \frac{v_\parallel}{v_\perp} E_{kz} J_n(\sigma) \qquad \sigma = k_\perp v_\perp / \Omega$$

$$\omega_{kr} - k_{\parallel} v_{\parallel} = \Omega \quad \longleftrightarrow \quad \omega_{kr} = \Omega + k_{\parallel} v_{\parallel}$$



- Alfvén/ion-cyclotron waves heat only counter-propagating protons!
- if $\omega/k_{\parallel} > 0$, then $v_{\parallel} < 0$.
- Since $\omega < \Omega$, the particle must propagate in the opposite direction as the wave, so that it sees a frequency that is Doppler-boosted up to Ω.

At low β, cyclotron heating results primarily in *perpendicular* heating



- $\beta_\mathrm{p} = \dfrac{8\pi n k_\mathrm{B} T}{B^2} = 2 \times \dfrac{4\pi n m_\mathrm{p}}{B^2} \times \dfrac{k_\mathrm{B} T}{m_\mathrm{p}} = 2\dfrac{v_\mathrm{th,p}^2}{v_\mathrm{A}^2}.$
- When $\beta_\mathrm{p} \ll 1$, $v_\mathrm{th,p} \ll v_\mathrm{A}$.
- Alfvén/ion-cyclotron waves satisfy $\dfrac{\omega}{k_\parallel} \sim v_\mathrm{A}$.
- $\longrightarrow$ resonant particles diffuse to larger $v_\perp$ and to slightly smaller $|v_\parallel|$.

Example 2: Self-Induced Scattering of Strahl Electrons

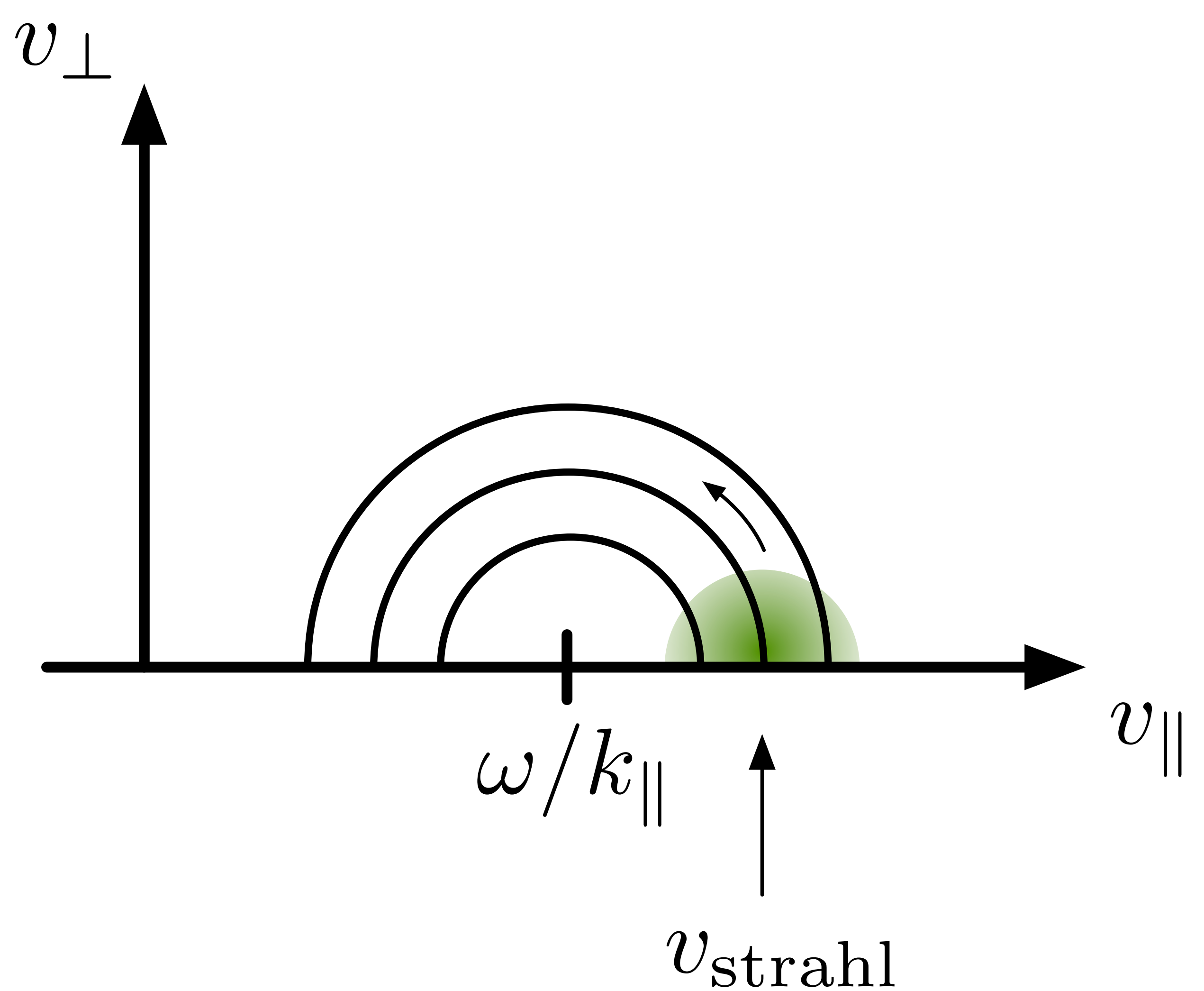
Verscharen, Chandran, Jeong, Salem, Pulupa, & Bale, ApJ, submitted. (arXiv:1906.02832v1)

- Why strahl electrons excite *oblique* whistlers, not *parallel-propagating whistlers.*

- Analytic instability criterion for strahl-excited whistlers in a low-beta plasma.

- Basic approach: use our *qualitative* understanding of quasilinear theory to determine the conditions under which whistler waves gain energy by interacting with an electron beam (the *strahl*) without losing energy to the core of the electron velocity distribution.

when particles lose energy by resonating with a wave, the wave gains energy and has a positive growth rate. Conversely, if the particles gain energy, the wave is damped.



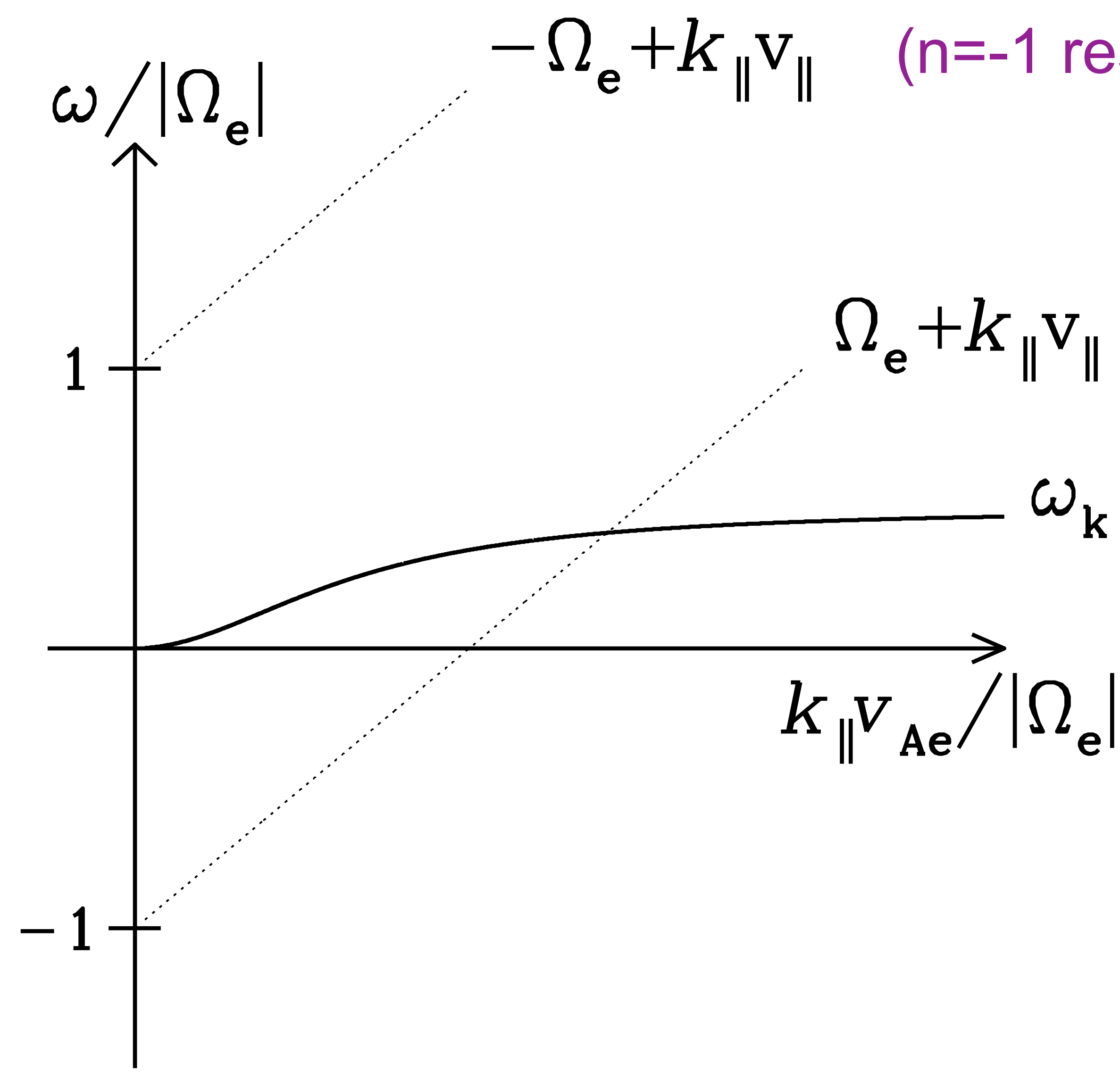
- When electrons resonate with a wave, they diffuse along arcs of constant energy in a frame moving along B_0 at speed $\omega/k_\parallel$.

- Instability requires $\boxed{0 < \omega/k_\parallel < v_{\mathrm{strahl}}}$

(Verscharen, Chandran, Jeong, Salem, Pulupa, & Bale, ApJ, submitted. (arXiv:1906.02832v1)

(Note that Ω_e is negative)

(Verscharen, Chandran, Jeong, Salem, Pulupa, & Bale, ApJ, submitted. (arXiv:1906.02832v1)



- the resonace condition is $\omega_k - k_\parallel v_\parallel = n\Omega_e$, or $\omega_k = n\Omega_e + k_\parallel v_\parallel$.
- Since the electrons driving the instability satisfy $0 < \omega_k/k_\parallel < v_{\text{strahl}}$, the instability is driven by the $n = +1$ resonance, not $n = -1$.
- ($n = 0$ won't work, because $\partial f/\partial v_\parallel < 0$ for solar-wind electrons, so electrons would gain energy from an $n = 0$ resonance, damping the wave.)

We consider whistlers at wavelengths much larger than the electron gyroradius. This means that σ << 1 below.

$$\frac{\partial f}{\partial t} = \lim_{V\to\infty} \sum_{n=-\infty}^{\infty} \frac{\pi q^2}{m^2} \int \frac{d^3k}{(2\pi)^3 V v_\perp} G\, v_\perp \delta(\omega_{kr} - k_\parallel v_\parallel - n\Omega) |\psi_{n,k}|^2 G\, f$$

$$G \equiv \left(1 - \frac{k_\parallel v_\parallel}{\omega_{kr}}\right) \frac{\partial}{\partial v_\perp} + \frac{k_\parallel v_\perp}{\omega_{kr}} \frac{\partial}{\partial v_\parallel}$$

As $\sigma \to 0$, $J_n(\sigma) \to 0$ unless $n = 0$.

$$\psi_{n,k} = \frac{1}{\sqrt{2}} \left[E_{k,r} e^{i\phi} J_{n+1}(\sigma) + E_{k,l} e^{-i\phi} J_{n-1}(\sigma)\right] + \frac{v_\parallel}{v_\perp} E_{kz} J_n(\sigma)$$

Since we're dominated by $n = +1$, only the left circularly polarized component of $\boldsymbol{E}$ contributes to the interaction. The parallel-propagating whistler, which is right circularly polarized, is not excited. We thus need oblique whistlers, with nonzero $k_\perp$, which are elliptically polarized and have a left circularly polarized component.

(Verscharen, Chandran, Jeong, Salem, Pulupa, & Bale, ApJ, submitted. (arXiv:1906.02832v1)

What is the Minimum Unstable Strahl Speed?

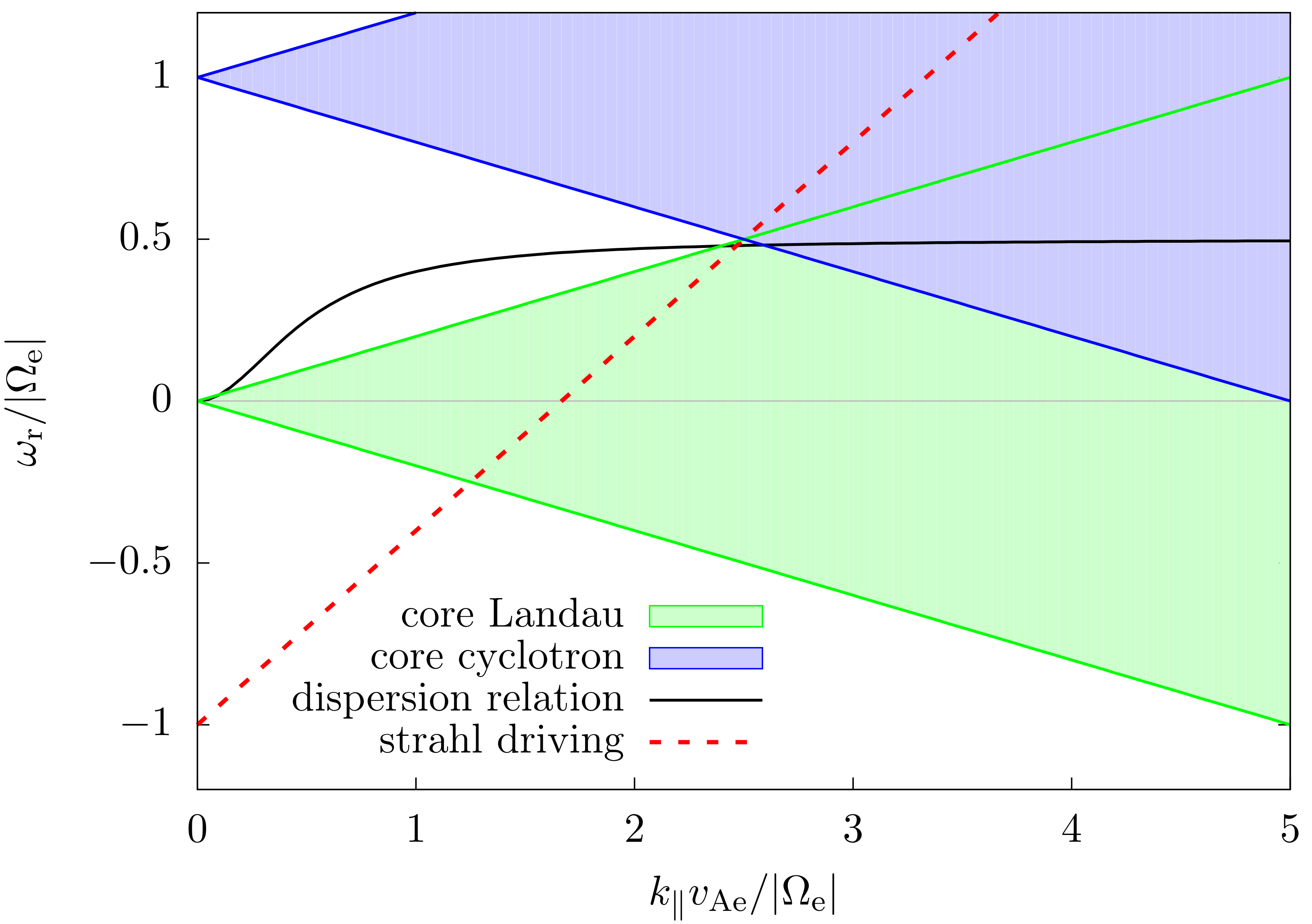


Figure 3. Dispersion relation and resonance conditions for the FM/W mode with $\theta = 60^\circ$ in regime 2. The black line shows Equation (14). The blue and green areas show Equations (16) and (17), respectively, and the red line shows Equation (13) with $U_s = 3w_c$. We use $w_c = 0.2v_{Ae}$. This situation represents a marginally stable state for the FM/W instability.

$$\frac{\omega}{|\Omega_e|} = \frac{(kd_e)^2\cos\theta}{1+(kd_e)^2}$$

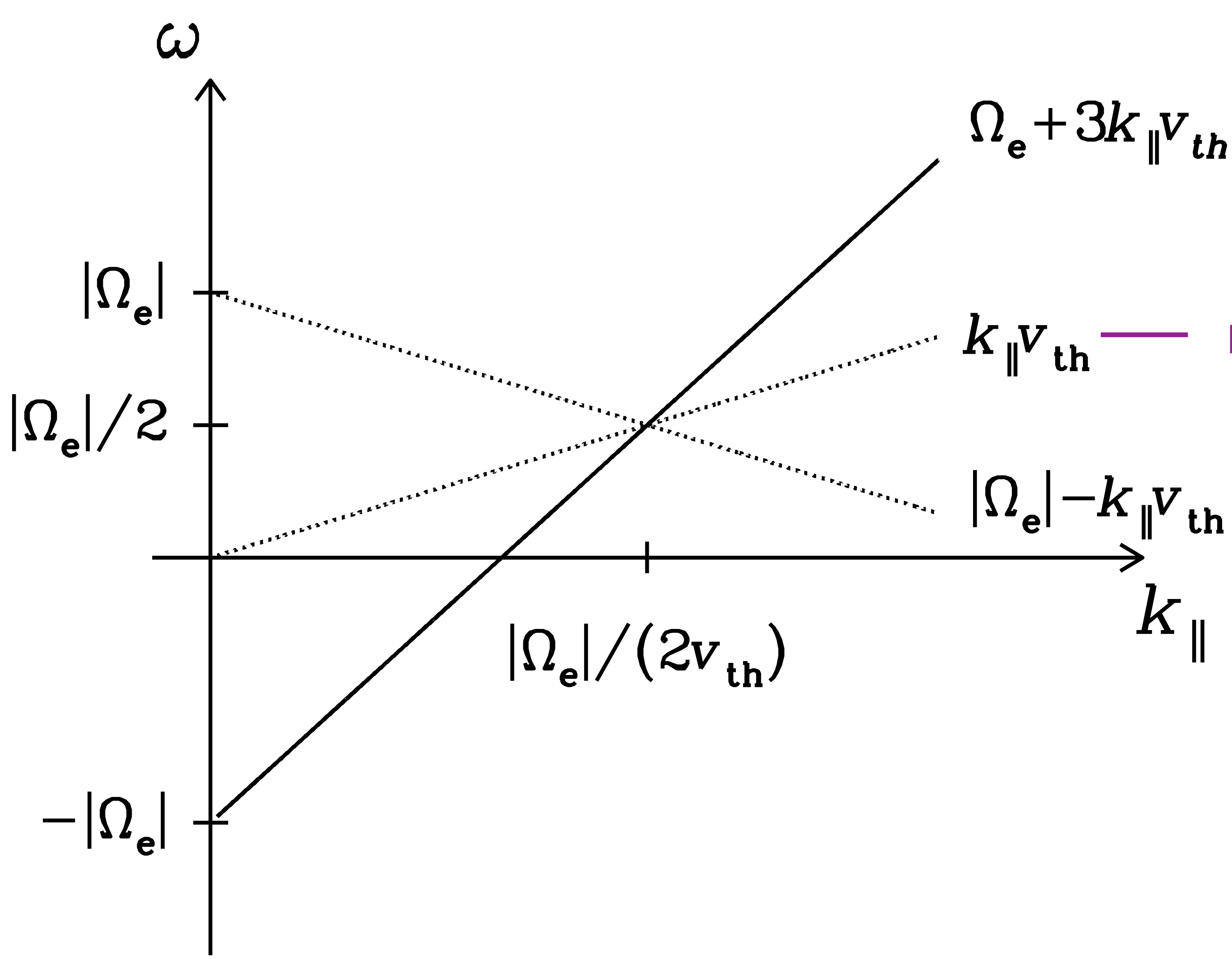
Waves in blue region undergo core cyclotron damping. Waves in green region undergo core Landau damping.

(Verscharen, Chandran, Jeong, Salem, Pulupa, & Bale, ApJ, submitted. (arXiv:1906.02832v1)

When Are Oblique Whistler Unstable?

(Verscharen, Chandran, Jeong, Salem, Pulupa, & Bale, ApJ, submitted. (arXiv:1906.02832v1)

When they cause the strahl to lose energy, but are not damped by thermal electrons in the core.



$\Omega_e + 3k_\parallel v_{th}$ —— strahl resonance line for $v_{||} = 3v_{th}$

$k_\parallel v_{th}$ —— resonance line for core Landau damping

$|\Omega_e| - k_\parallel v_{th}$ —— resonance line for core cyclotron damping

$$\boxed{\text{Need } v_{\rm strahl} \gtrsim 3v_{\rm th}}$$

(but this only works when beta is small, otherwise the dispersion relation is in the Landau-damped region.)

Parenthetical comment: in case you're interested, at higher β, the whistlers are Landau damped, and the instability threshold on the strahl speed increases.

Verscharen, Chandran, Jeong, Salem, Pulupa, & Bale, ApJ, submitted. (arXiv:1906.02832v1)

$$U_{\mathrm{s,min}} = \left[\frac{2n_{\mathrm{c}} v_{\mathrm{th,c}} v_{\mathrm{th,s}} v_{\mathrm{Ae}}^2 (1+\cos\theta)}{n_{\mathrm{s}}(1-\cos\theta)\cos\theta}\right]^{1/4}$$

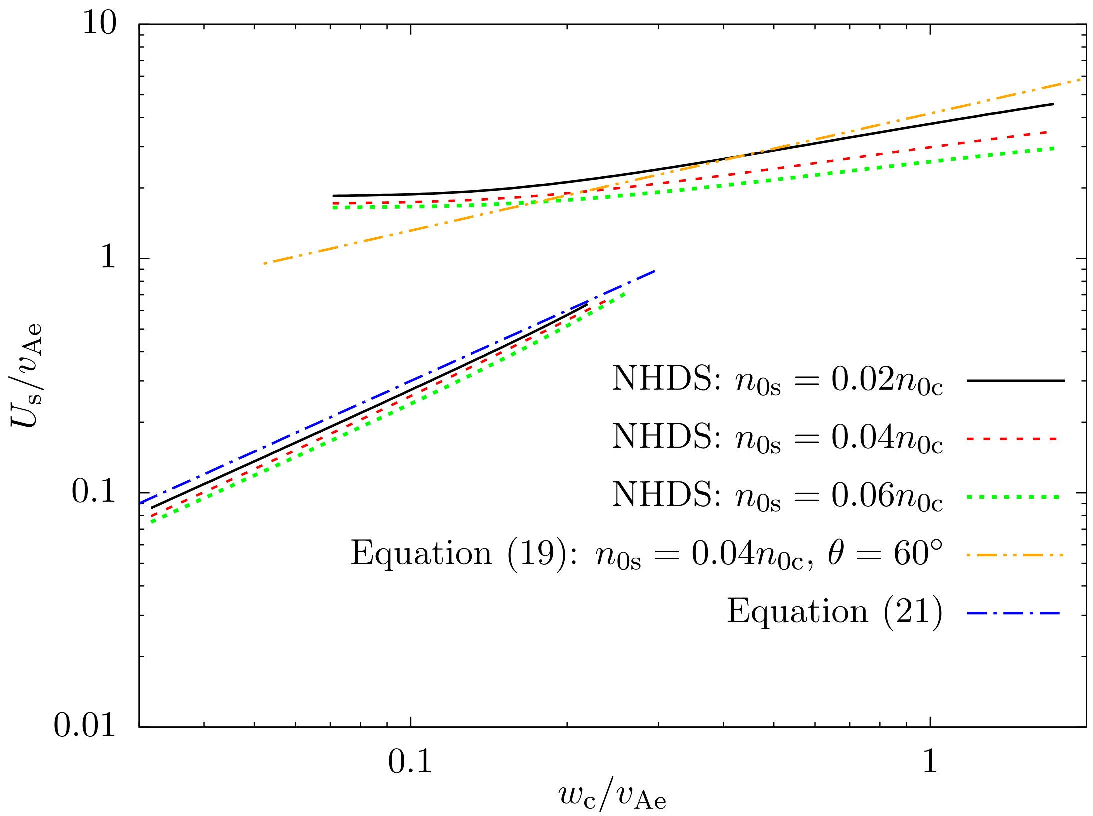


Figure 4. Comparison of Equations (19) and (21) with numerical solutions of the hot-plasma dispersion relation from our NHDS code. The orange and blue lines show Equations (19) and (21), except that the $\gtrsim$-signs have been replaced with equal-signs. We use $w_{\mathrm{s}} = 2w_{\mathrm{c}}$ and $T_{0\mathrm{p}} = T_{0\mathrm{c}}$. For the numerical solutions, we show isocontours of constant maximum growth $\gamma_k = 10^{-3}|\Omega_{\mathrm{e}}|$ with $v_{\mathrm{Ap}}/c = 10^{-4}$. The analytical solutions use $\theta = 60°$, while the numerical solutions are evaluated at the angle for which the lowest U_{s} leads to a maximum growth rate of $\gamma = 10^{-3}|\Omega_{\mathrm{e}}|$.

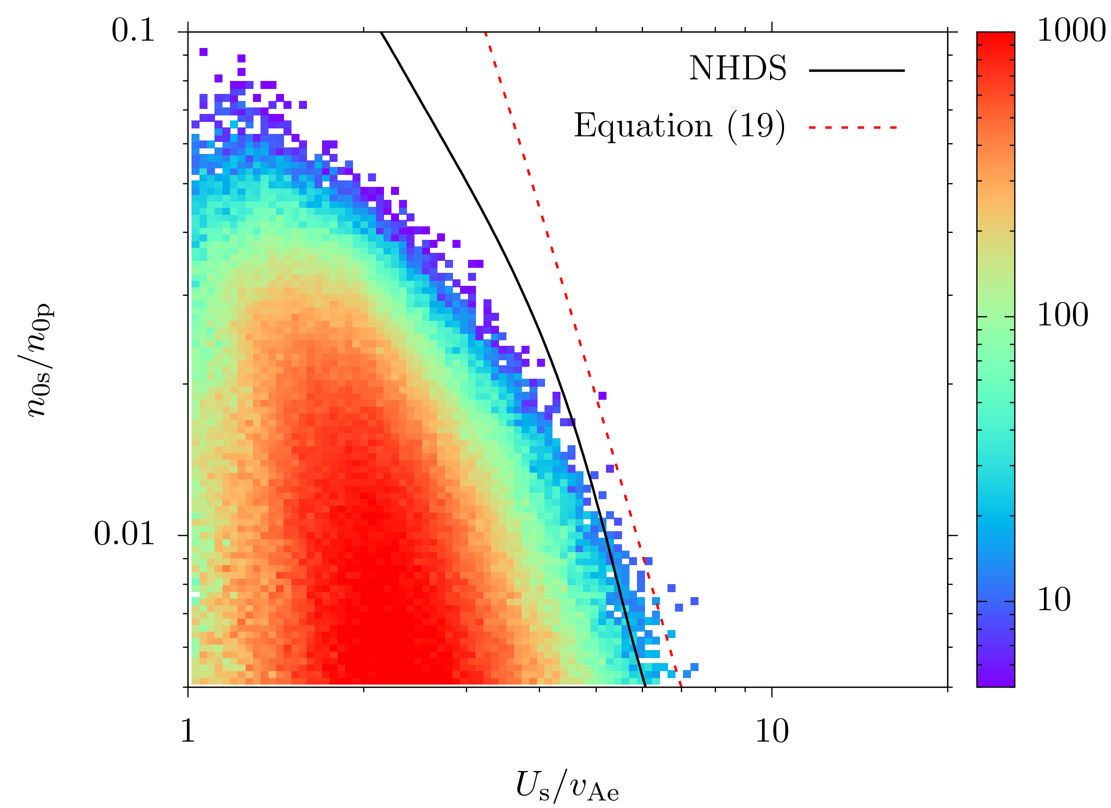


Figure 6. Data distribution of the analyzed solar-wind interval in the $n_{0\mathrm{s}}/n_{0\mathrm{p}}$ vs. $U_{\mathrm{s}}/v_{\mathrm{Ae}}$ plane. The color-coding shows the probability density in the corresponding bin in arbitrary units. The black line shows the isocontour of maximum growth rate $\gamma_{\mathrm{m}} = 10^{-3}|\Omega_{\mathrm{e}}|$ for the oblique FM/W instability from our NHDS solutions. The red dashed line shows Equation (19) for $\theta = 60°$, and $w_{\mathrm{c}} = v_{\mathrm{Ae}} = w_{\mathrm{s}}$.

Other examples: deceleration of alpha-particle beams by fast-magnetosonic waves, cosmic-ray streaming instability, limits on proton temperature anisotropy from firehose, mirror, and cyclotron waves, …

Outline

√ I. The Lost Art of Quasilinear Theory.

A. ion cyclotron heating

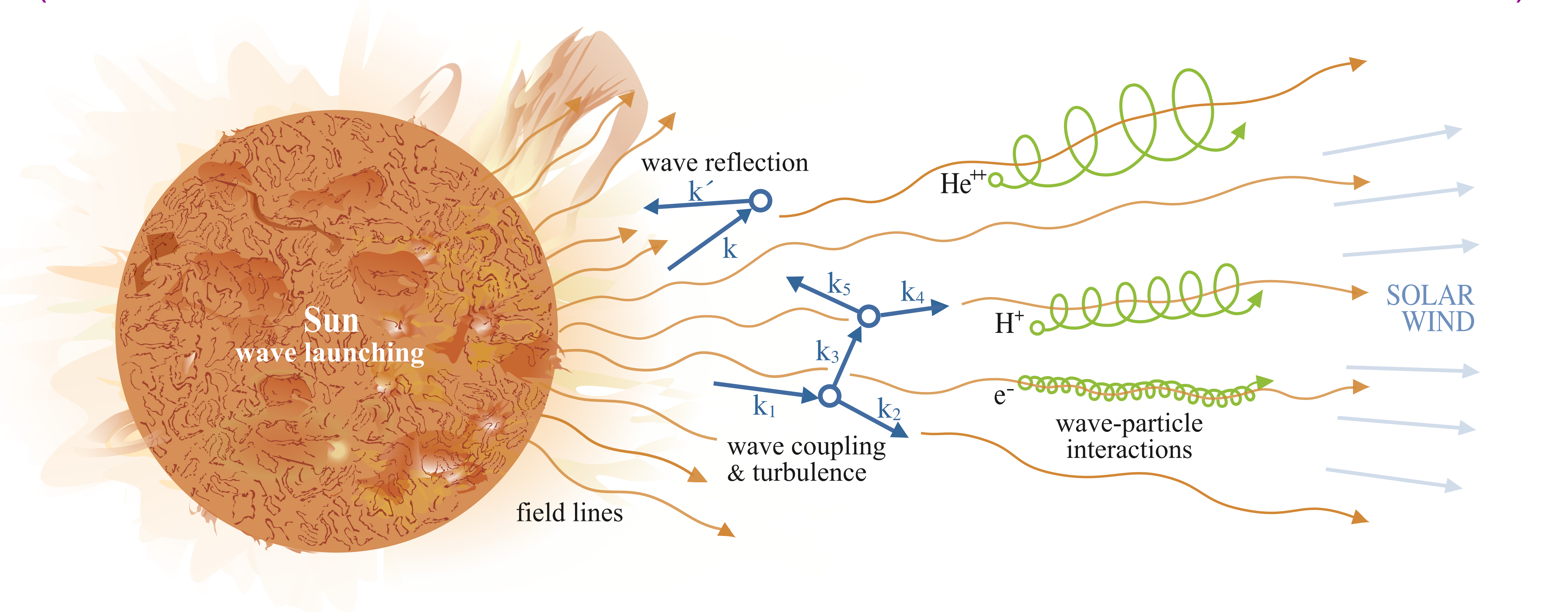
B. self-induced scattering of the electron *strahl* by oblique whistler waves

C. (deceleration of alpha-particle beams, limits on proton temperature anisotropy)

II. Dissipation of Solar-Wind Turbulence by Non-Resonant Stochastic Ion Heating

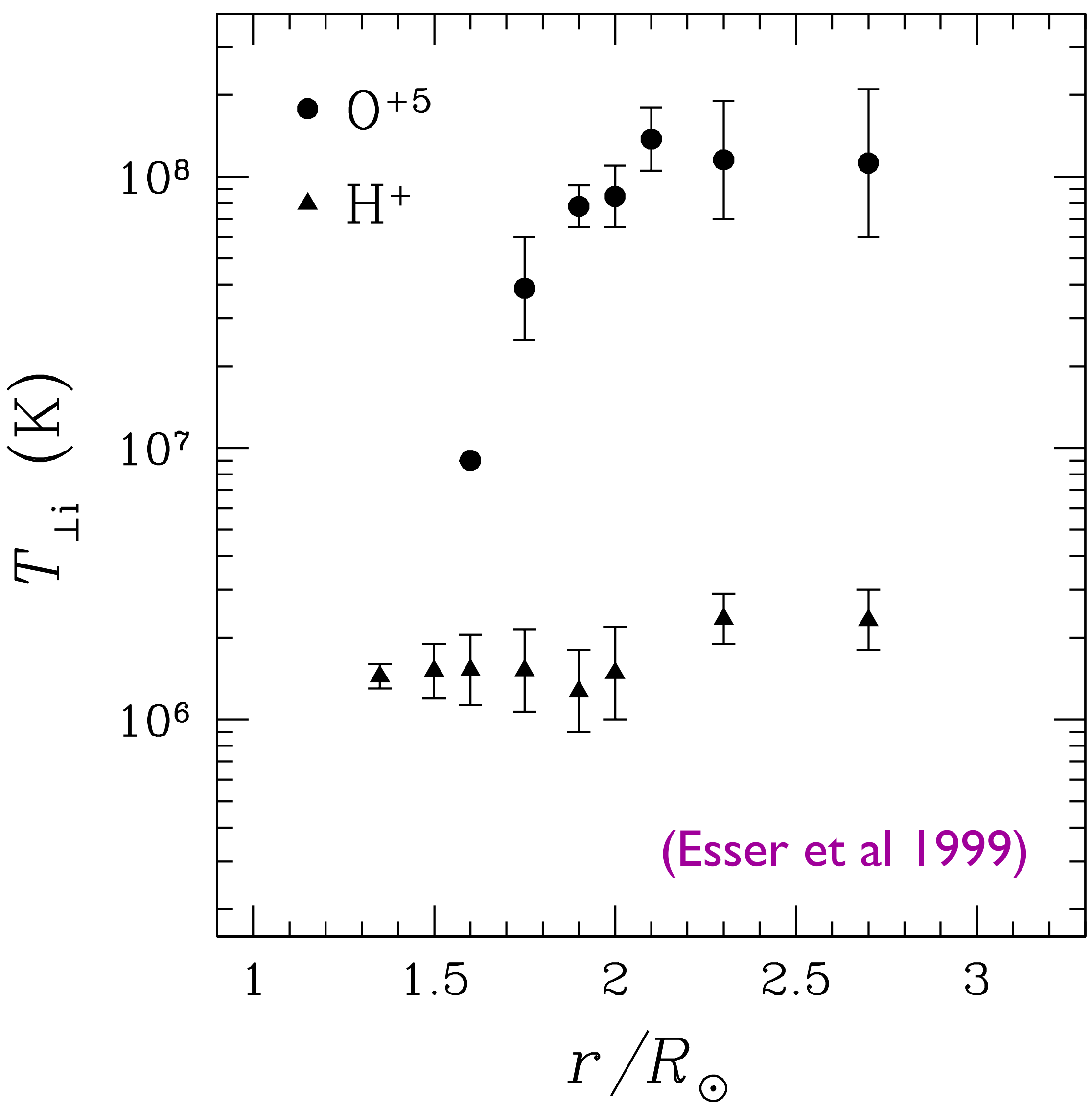
Coronal Heating and Solar-Wind Acceleration by Waves

(Parker 1965, Coleman 1968, Velli et al 1989, Zhou & Matthaeus 1989, Cranmer et al 2007)



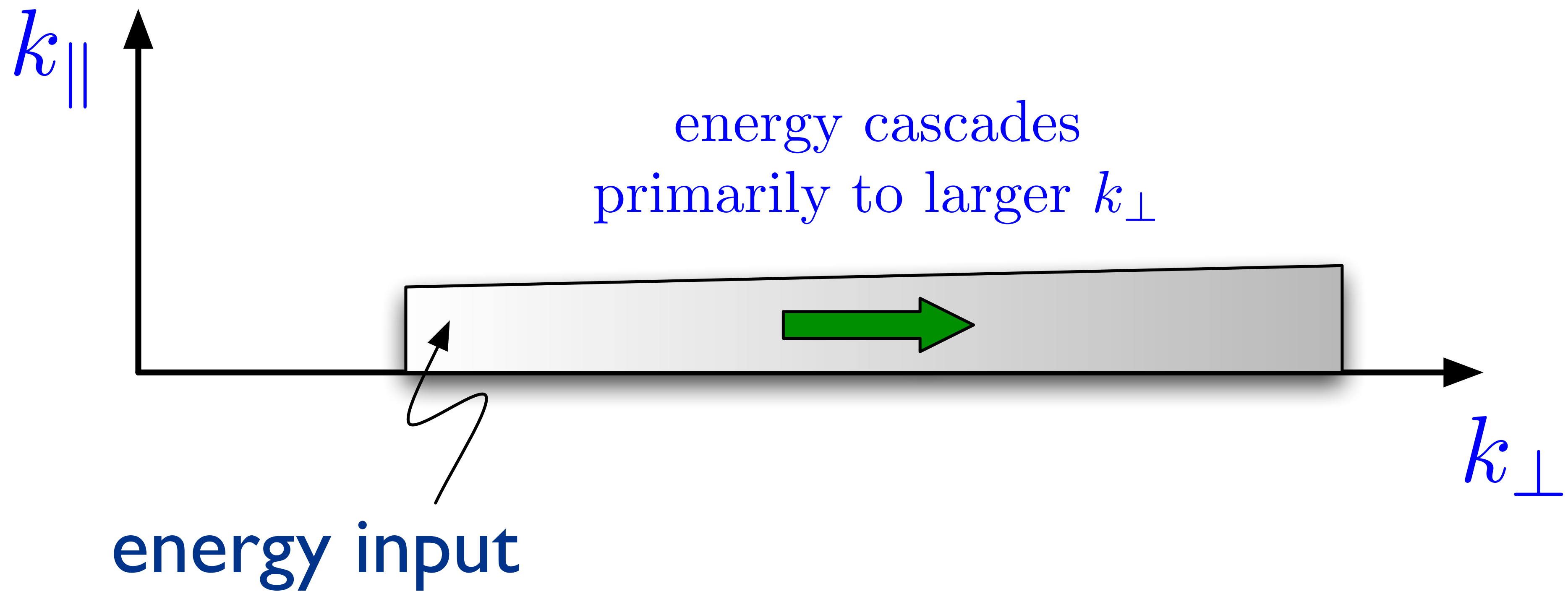
- The Sun launches Alfven waves, which transport energy outwards
- The waves become turbulent, which causes wave energy to 'cascade' from long wavelengths to short wavelengths
- Short-wavelength waves dissipate, heating the plasma. This increases the thermal pressure, which, along with the wave pressure, accelerates the solar wind.

Key Problem: Can Turbulence Explain the *Perpendicular* Ion Heating Observed in the Corona?



- These are *perpendicular* temperatures inferred from line widths observed at the Sun's limb.
- Protons in the corona and low-β fast-solar-wind streams satisfy $T_\perp > T_{||}$

Stochastic Heating by Strong Alfvén-Wave (AW) and kinetic-Alfvén-wave (KAW) Turbulence



Because the AW frequency is $\omega = k_\parallel v_{\rm A}$, the small-scale AWs produced by the cascade have low frequencies

Does the dissipation of low-frequency, strong AW/KAW turbulence cause "perpendicular" ion heating, and if so, how?

Magnetic Moment Conservation

- If an ion's orbit is nearly periodic in the plane perpendicular to B, and if the frequencies of the fluctuating electric and magnetic fields are much smaller than the ion's cyclotron frequency, then the ion's magnetic moment μ is almost exactly conserved (Kruskal 1962), where

$$\mu = \frac{mv_\perp^2}{2B}$$

- Possible route to perpendicular heating from low-frequency AW turbulence: if the gyro-scale fluctuations are large enough, then an ion's orbit becomes "stochastic," and μ is not conserved. (McChesney, Stern, & Bellan 1987; Johnson & Cheng 2001; Chen, Lin, & White 2001; Chaston et al 2003; Voitenko & Goosens 2004)

Criterion for Stochasticity in Low-β Plasmas

Let δv_ρ and δB_ρ be the rms amplitudes of the velocity and magnetic field fluctuations at $k_\perp \rho_i \approx 1$.

Stochasticity Criterion (McChesney, Stern, & Bellan 1987; Chaston et al 2004; Chandran et al 2010):

$$\varepsilon \equiv \delta v_\rho / v_\perp \sim O(1)$$

Implies that the fractional change in an ion's K.E. during a single gyro-orbit is of order unity.

$$\text{For protons,} \quad \frac{\delta v_\rho}{v_\perp} = \frac{v_\mathrm{A}}{v_\perp} \cdot \frac{\delta v_\rho}{v_\mathrm{A}} \simeq \beta^{-1/2} \cdot \frac{\delta B_\rho}{B_0}$$

To achieve the heating rate in the corona, does $\delta v_\rho / v_\perp$ need to be = 1, or is a smaller value sufficient?

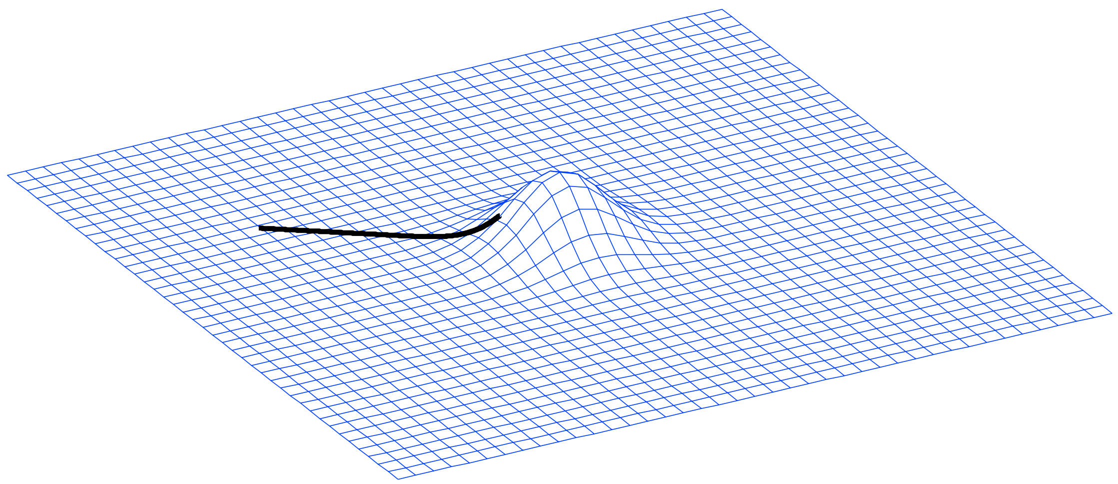
What Physical Process Energizes the Ions?

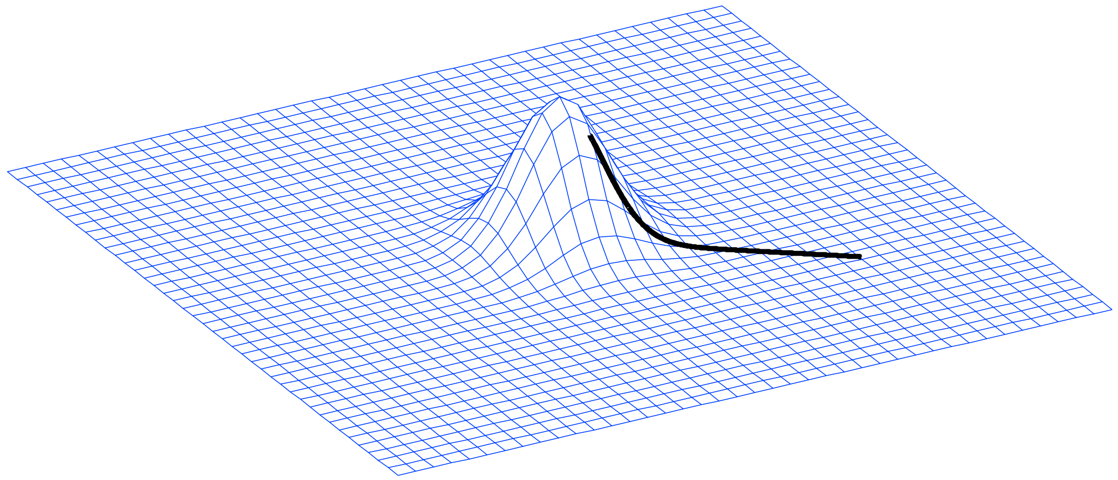
$$H = q\Phi + \frac{1}{2m}\left(\mathbf{p} - \frac{q}{c}\mathbf{A}\right)^2$$

$$H = q\Phi + \frac{mv^2}{2}$$

$$\frac{dH}{dt} = q\frac{\partial\Phi}{\partial t} - \frac{q\mathbf{v}}{c}\cdot\frac{\partial\mathbf{A}}{\partial t}$$

$$\frac{dH}{dt} \simeq q\frac{\partial\Phi}{\partial t}$$





- When a particle "rolls over" a rising "potential-energy hill," the hill is shorter when the particle rolls up and higher when the particle rolls down, so the particle gains kinetic energy.

Stochastic Heating by AWs, KAWs, or Strong RMHD/ KAW Turbulence

(Chandran, Li, Rogers, Quataert, & Germaschewski, ApJ, 720, 503, 2010.

Much earlier related work by, e.g., McChesney et al 1987, Karimabadi et al 1994, Chen et al 2001, Johnson & Cheng 2001, Chaston et al 2004, Fiksel et al 2009)

Particles diffuse in both space and energy. Derivation based on phenomenological arguments at $\boxed{\beta \lesssim 1}$ leads to:

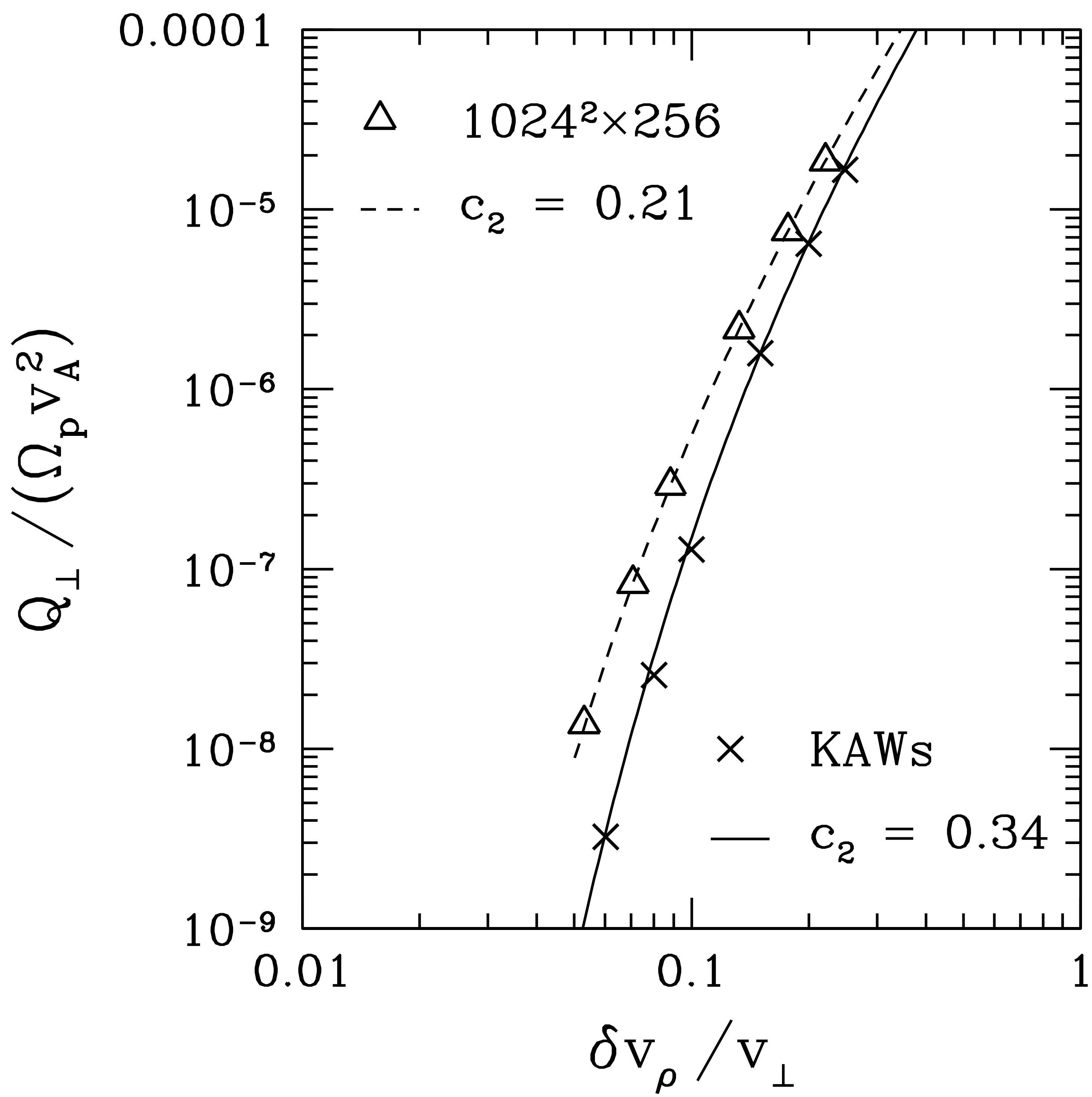
$$Q_\perp = \frac{c_1 (\delta v_\rho)^3}{\rho} \exp\left(-\frac{c_2}{\epsilon}\right)$$

Here ρ is the ion gyroradius, δv_ρ is the rms velocity at scale ρ, and $\varepsilon = \delta v_\rho / v_\perp$. The dimensionless constants c_1 and c_2 depend on whether the fluctuations are waves or turbulence and on the degree of intermittency.

Numerical Simulations of Test-Particle Protons Interacting with Either KAWS or Strong RMHD Turbulence.

(RMHD turbulence: Xia, Perez, Chandran, & Quataert 2013)

(KAWs: Chandran, Li, Rogers, Quataert, & Germaschewski, ApJ, 720, 503, 2010)



Important Point: Stochastic Heating is Inherently Self-Limiting at Low Beta

(Chandran 2010)

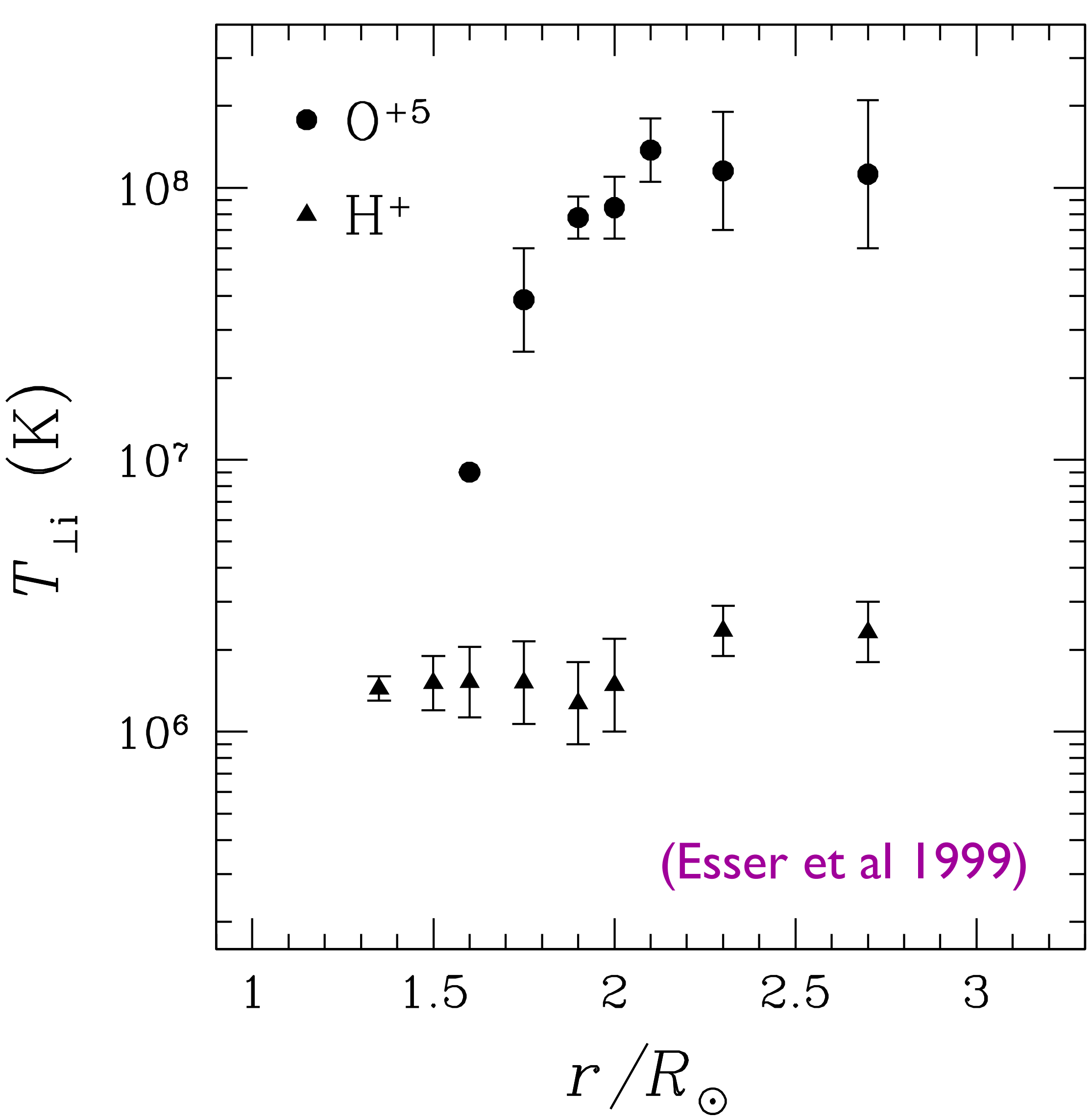
$$Q_\perp = \frac{0.74(\delta v_\rho)^3}{\rho} \exp\left(-\frac{0.21}{\epsilon}\right)$$

- As $T_\perp$ increases,

  $\varepsilon = \delta v_\rho / v_\perp$ decreases,

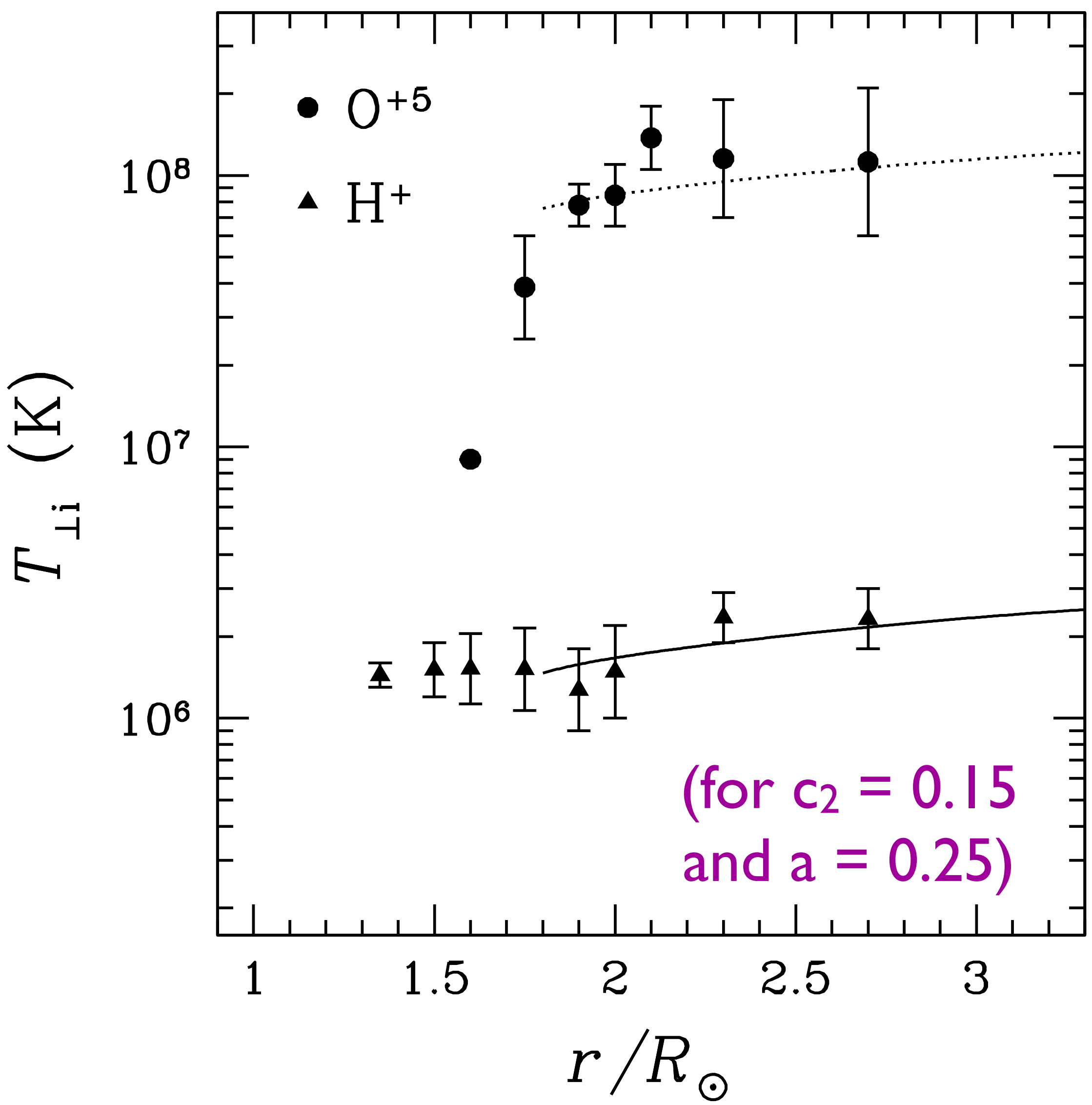
  and $Q_\perp$ decreases

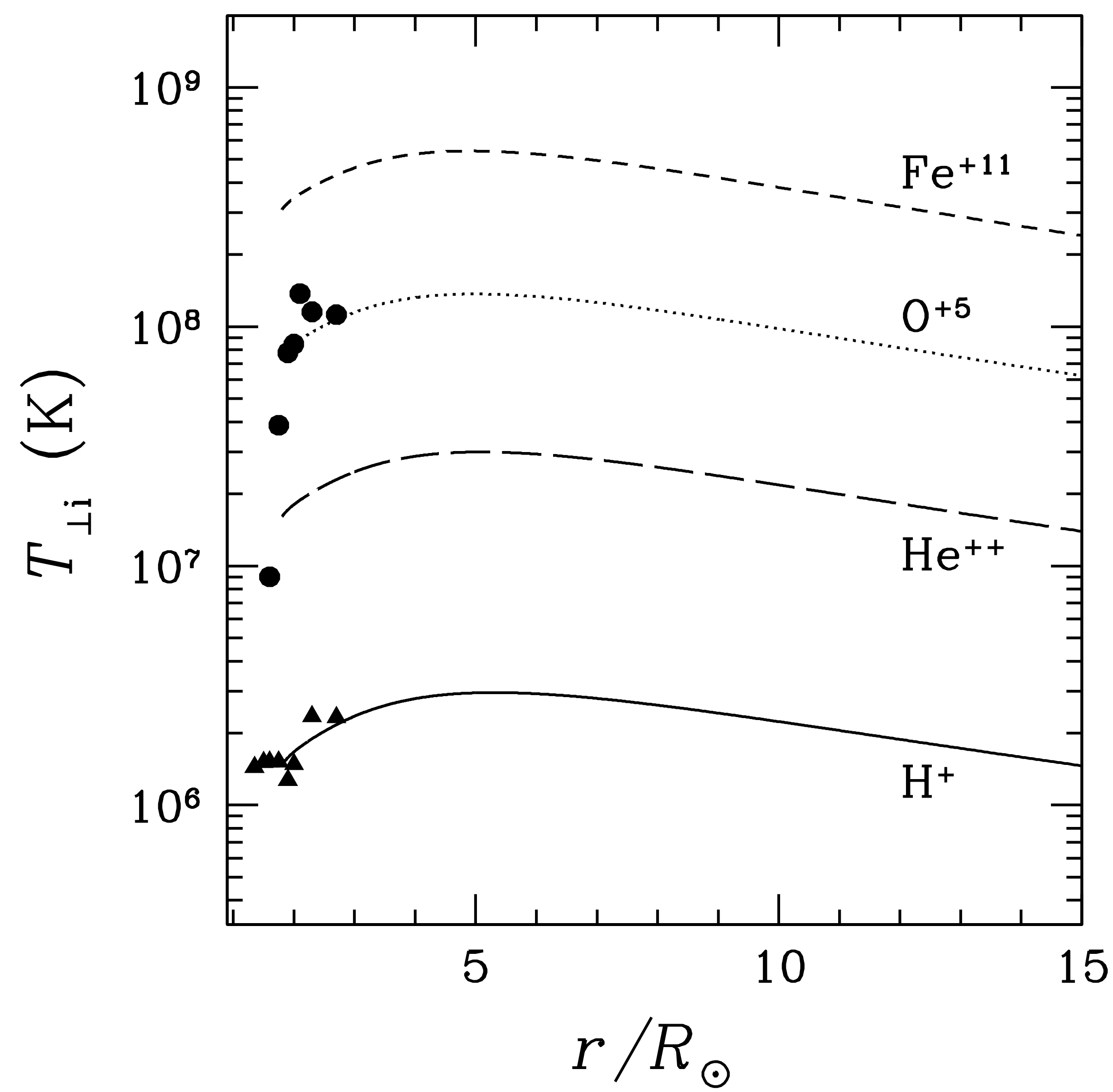
  as a result.



Ion Temperature Profiles from Stochastic Heating

(Chandran 2010)

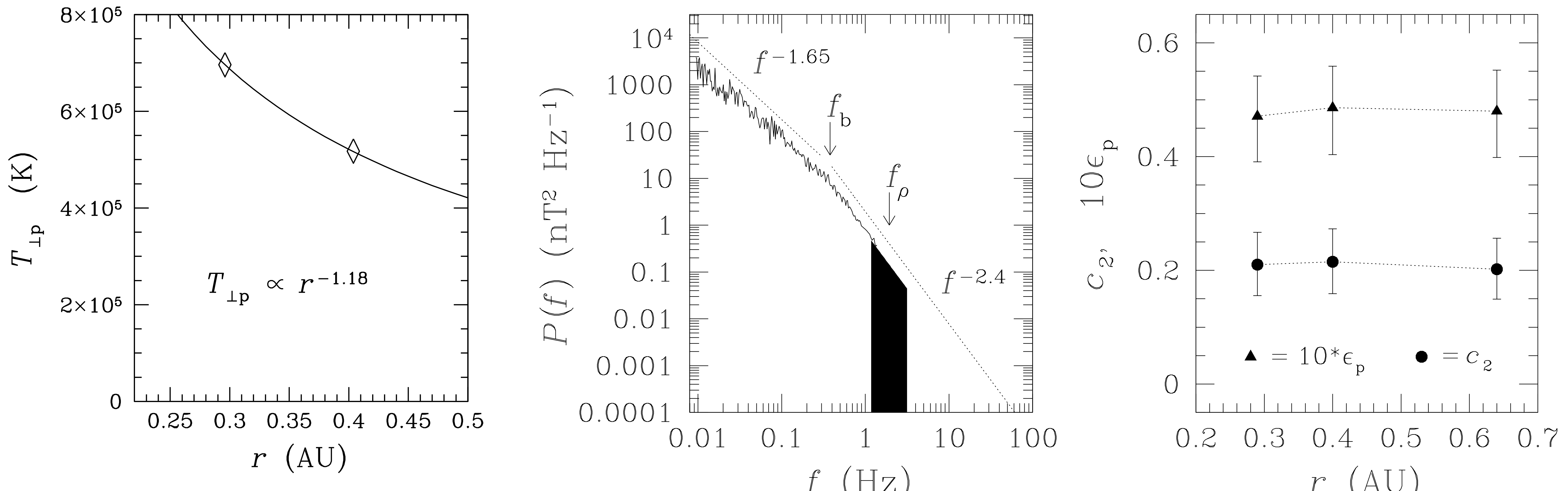




$$T_{\perp \mathrm{i}} = \frac{m_{\mathrm{i}}}{2k_{\mathrm{B}}} \left[\frac{\alpha_{\mathrm{i}} \delta v_0}{\varepsilon_{\mathrm{i}} (L_\perp \Omega_{\mathrm{i}})^a} \right]^{2/(1-a)}$$

Observational Test of Stochastic Proton Heating at 0.3 AU to 0.64 AU

(Bourouaine & Chandran 2013)



$$Q_{\perp \mathrm{empirical}} = \frac{U}{r^2}\frac{d}{dr}\left(\frac{r^2 k_B T_\perp}{m_p}\right) \qquad Q_{\perp \mathrm{stoch}} = \frac{c_1 (\delta v_p)^3}{\rho_p} \exp\left(-\frac{c_2}{\epsilon_p}\right)$$

- Left panel: we evaluate an "empirical" perpendicular heating rate from the measured values of U and $T_\perp(r)$ in *Helios* data for the fast solar wind (Marsch et al 1983).
- We use *Helios* data to measure δB_p (middle panel), set $\delta v_p = \sigma v_A \delta B_p / B_0$, and use this value of δv_p to determine the stochastic heating rate, with σ=1.19.
- We then find the values of c_1 and c_2 for which $Q_{\perp \mathrm{empirical}} = Q_{\perp \mathrm{stoch}}$ (right panel). Lower error bar corresponds to σ=1, and upper error bar corresponds to σ=1.38.

Conclusion

- Quasilinear theory is a powerful tool for understanding resonant wave-particle interactions and involves three organizing principles:

    1. resonance condition: $\omega - k_\parallel v_\parallel = n\Omega$
    2. particle energy is conserved in the wave frame
    3. right (left) circularly polarized waves at $k_\perp \rho \ll 1$ interact only through the $n = -1$ ($n = 1$) resonance.

- At a conceptual level, quasilinear theory can be used to deduce important properties of wave-particle interactions, including the following:

    – parallel-propagating Alfvén/ion-cyclotron waves interact only with counter-propagating protons and at low β cause primarily perpendicular ion heating

    – the electron *strahl* excites primarily *oblique* whistler waves, which at low β become unstable when $V_{\rm strahl} \gtrsim 3 v_{\rm th,e}$.

- Strong Alfvén-wave/kinetic-Alfvén-wave turbulence causes perpendicular ion heating through a non-resonant process called stochastic ion heating.

- Simulations are a valuable tool, but even if you are a computational expert, study theory carefully, because it provides crucial insights into numerical and experimental data.