Kinetic Theory

Amitava Bhattacharjee
Princeton Plasma Physics Laboratory,
Princeton University

2019 Heliophysics Summer School

Goal of this lecture

- Review a few basic plasma concepts in plasma kinetic theory that underlie the lectures later in the week.
- There are several excellent text books: Nicholson (out of print), Goldston and Rutherford, Boyd and Sanderson.
- The book I am most familiar with is by Gurnett and Bhattacharjee, from which most of the material is taken.

Plasma: levels of description

Plasma is an ensemble of charged particles, capable of exhibiting collective interactions.

Levels of Description:

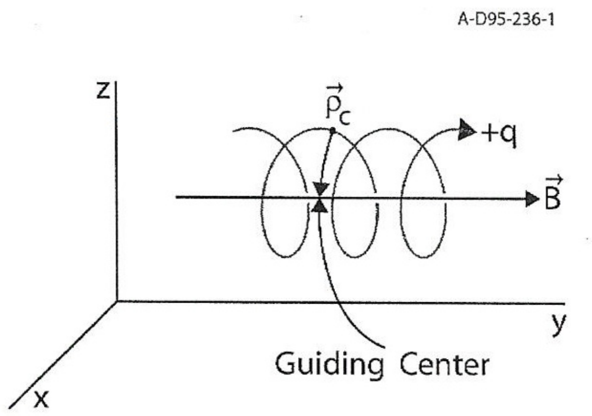
- Single-particle dynamics in *prescribed* electric and magnetic fields
- Plasmas as fluids in 3D configuration space moving under the influence of *self-consistent* electric and magnetic fields
- Plasmas as kinetic fluids in 6D μ-space (that is, configuration and velocity space), coupled to *self-consistent* Maxwell's equations.

Single-Particle Orbit Theory

Newton's law of motion for charged particles

$$m\frac{d\mathbf{v}}{dt} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$$

Guiding-Center: A very useful concept



Single-Particle Orbit Theory

ExB Drift

$$m\frac{d\mathbf{v}}{dt} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$$

Consider $\mathbf{E} = const.,\ \mathbf{B} = const.$

The charged particles experience a drift velocity, perpendicular to both E and B, and independent of their charge and mass.

$$\mathbf{v}_{\mathbf{E}} = \frac{\mathbf{E} \times \mathbf{B}}{B^2}$$

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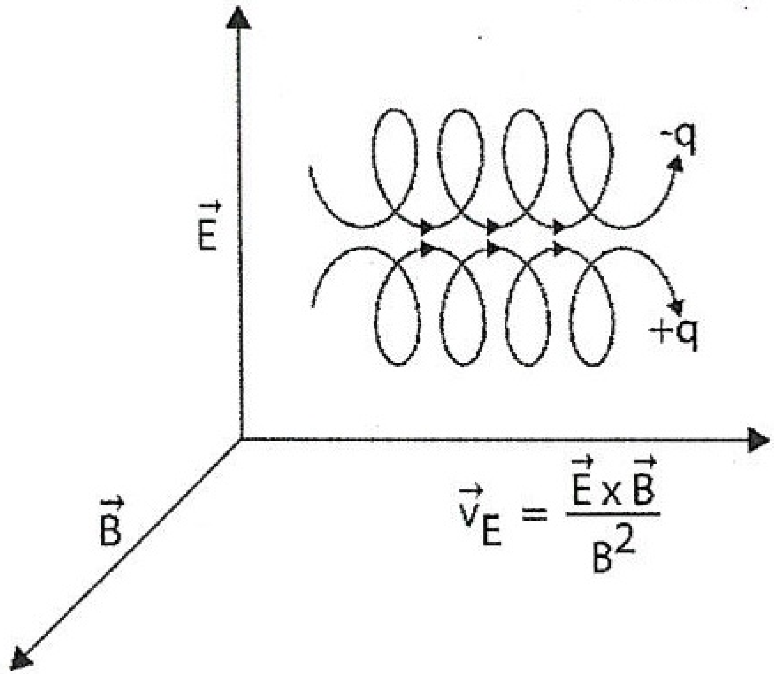

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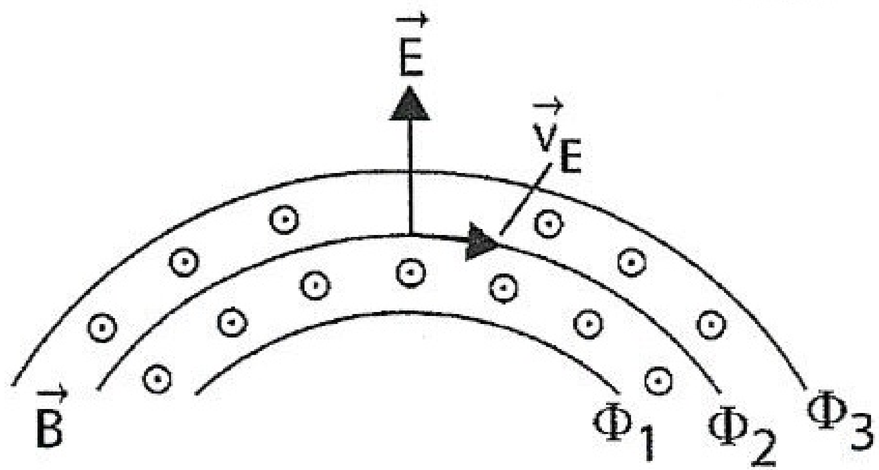
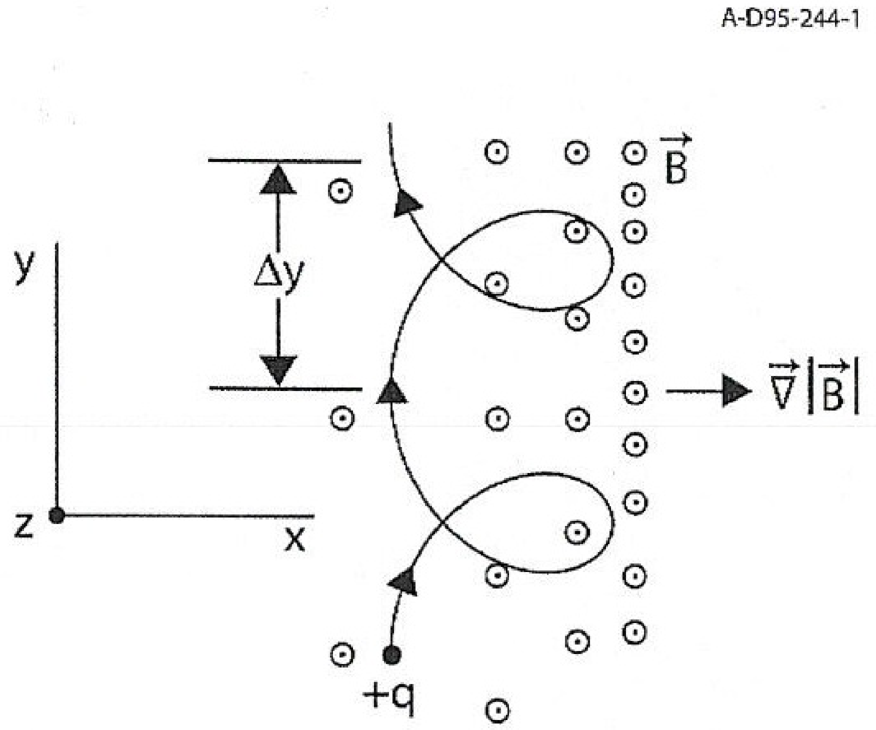


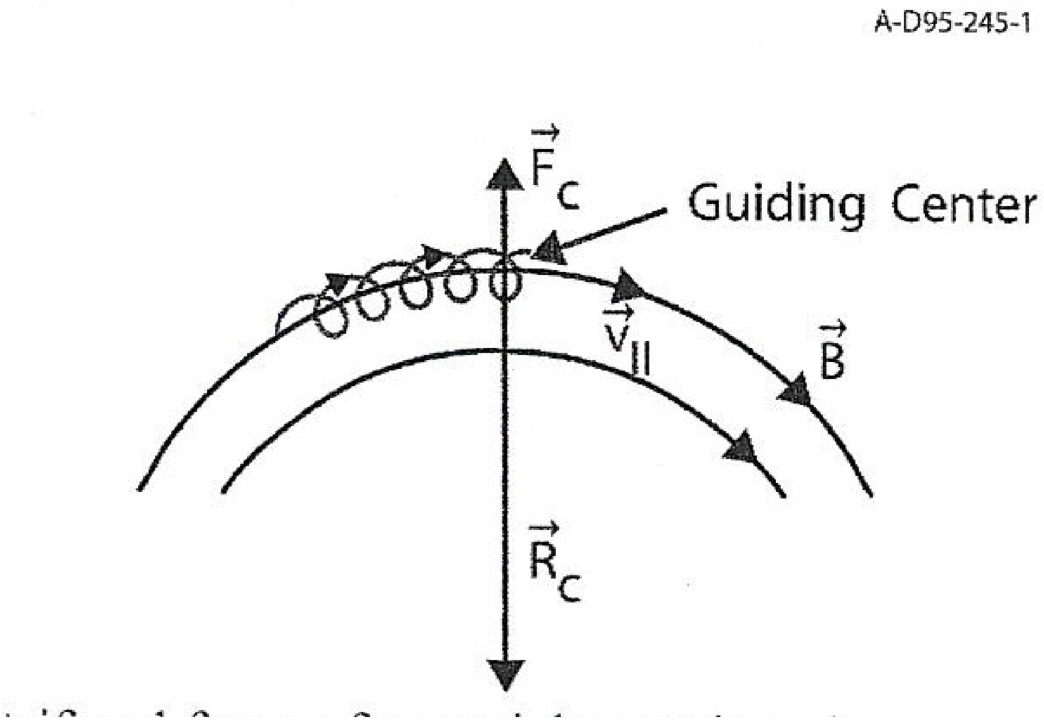
Gradient B drift



$$\mathbf{V_G} = \frac{w_\perp}{qB}\left(\frac{\mathbf{B}\times\nabla\mathbf{B}}{B^2}\right),$$

$$w_\perp = \frac{1}{2}\omega_c{}^2\rho_c{}^2$$

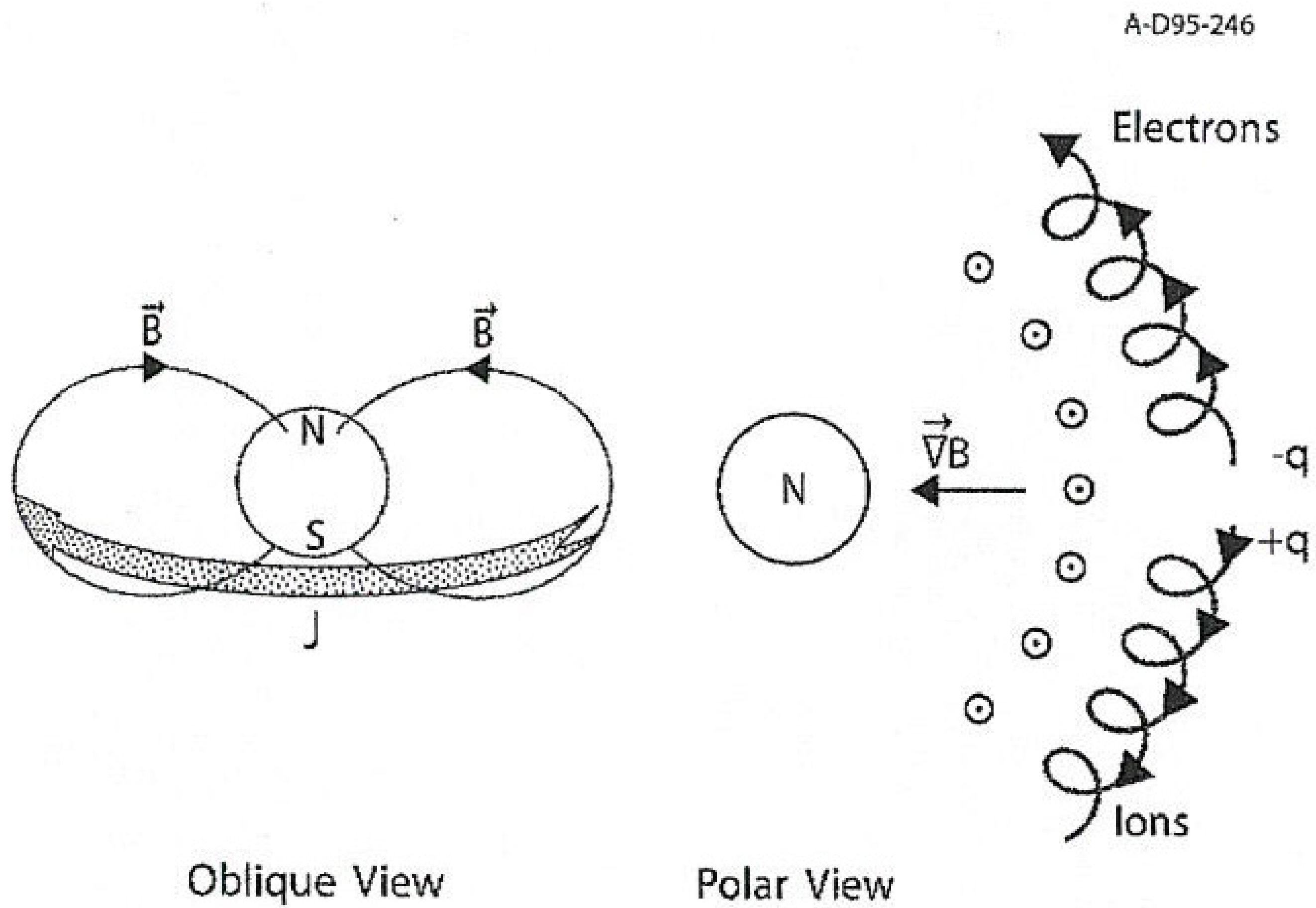
Curvature drift



$$\mathbf{V}_C = \frac{2w_{\|}}{qB^2}\left(\frac{\mathbf{R_C} \times \mathbf{B}}{R_C{}^2}\right),$$

$$w_{\|} = \frac{1}{2} m v_{\|}{}^2$$

The Ring Current in Earth's Magnetosphere: An Example



Kinetic Description of Plasmas

Distribution function $f(\mathbf{r},\mathbf{v},t)$

Total number of particles $N = \iint\limits_{phase\ space} d\mathbf{x}\, d\mathbf{v} f(\mathbf{r},\mathbf{v},t)$

Example: Maxwell distribution function

$$f = n_0 \exp\left(-\frac{mv^2}{2kT}\right),\ n_0 = N/V$$

Boltzmann-Vlasov Equation

Motion of an incompressible phase fluid in μ–space (6D)

$$\frac{\partial f_s}{\partial t} + \mathbf{v}.\frac{\partial f_s}{\partial t} + \frac{q}{m}\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right).\frac{\partial f_s}{\partial \mathbf{v}} = 0,\ s = e,i$$

In the presence of collisions

$$\frac{\partial f_s}{\partial t} + \mathbf{v}.\frac{\partial f_s}{\partial t} + \frac{q}{m}\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right).\frac{\partial f_s}{\partial \mathbf{v}} = (\frac{\delta f_s}{\delta t})_c$$

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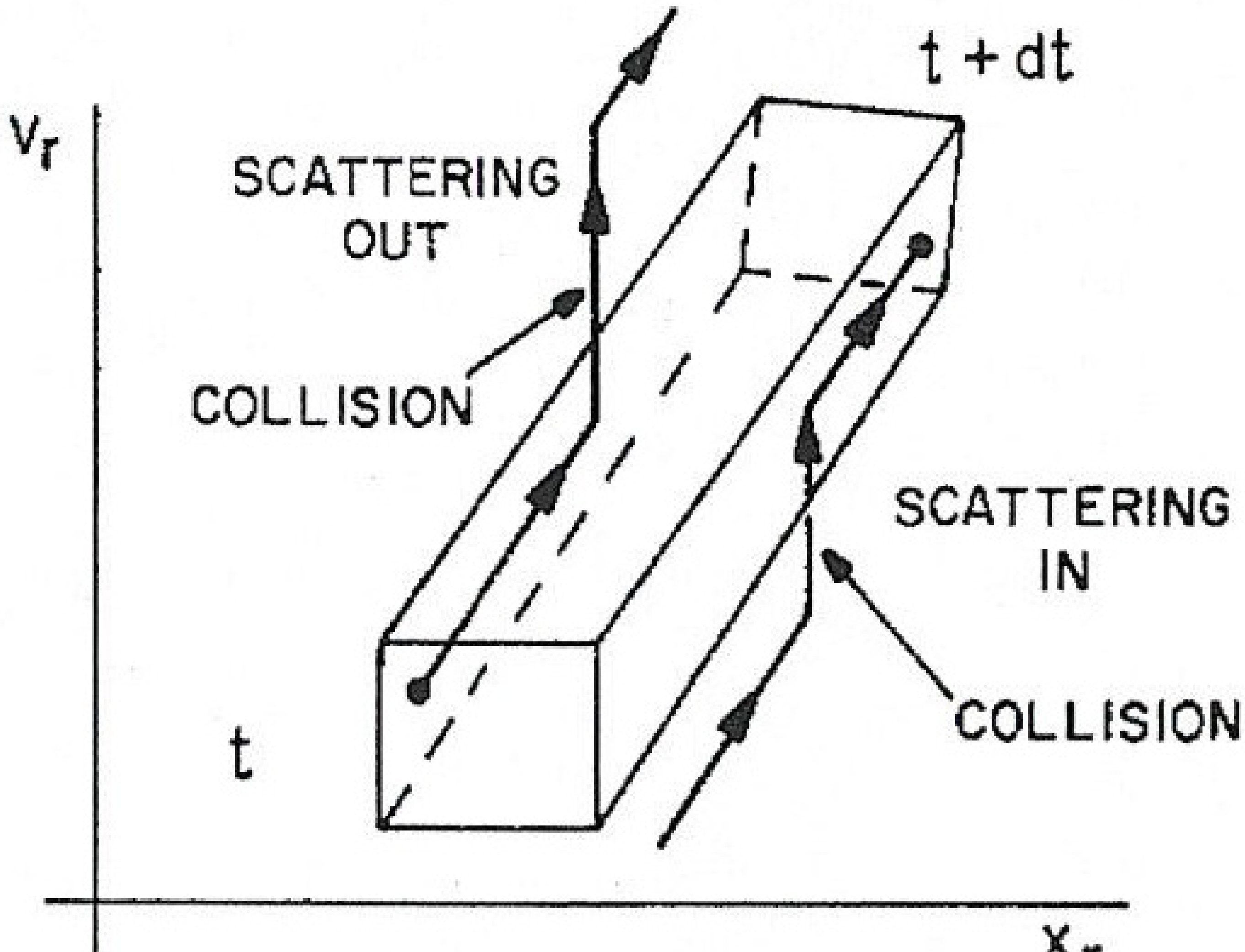

t + dt
V_r
SCATTERING
OUT
COLLISION
SCATTERING
IN
COLLISION
t
X_r

Properties of the Vlasov Equation

1. The Vlasov equation conserves the total number of particles N of a species, which can be proven, for the one-dimensional case, as follows:

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial t} \iint f \, \mathrm{d}x\mathrm{d}v = -\iint v \frac{\partial f}{\partial x} \mathrm{d}x\mathrm{d}v - \iint a \frac{\partial f}{\partial v} \mathrm{d}x\mathrm{d}v$$

2. Any function, $g[\frac{1}{2}mv^2 + q\Phi(x)]$, which can be written in terms of the total energy of the particle, is a solution of the Vlasov equation.

3. The Vlasov equation has the property that the phase-space density f is constant along the trajectory of a test particle that moves in the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$. Let $[\mathbf{x}(t), \mathbf{v}(t)]$ be the trajectory that follows from the equation of motion $m\dot{\mathbf{v}} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ and $\dot{\mathbf{x}} = \mathbf{v}$, then

$$\begin{aligned}\frac{\mathrm{d}f(\mathbf{x}(t), \mathbf{v}(t), t)}{\mathrm{d}t} &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \mathbf{x}} \cdot \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} + \frac{\partial f}{\partial \mathbf{v}} \cdot \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \\ &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \mathbf{x}} \cdot \mathbf{v} + \frac{\partial f}{\partial \mathbf{v}} \cdot \frac{q}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = 0\,.\end{aligned}$$

Properties of the Vlasov Equation

4. The Vlasov equation is invariant under time reversal, $(t \to -t)$, $(\mathbf{v} \to -\mathbf{v})$. This means that there is no change in entropy for a Vlasov system.

$$\frac{\partial f_s}{\partial t} + \mathbf{v}.\frac{\partial f_s}{\partial t} + \frac{q}{m}\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right).\frac{\partial f_s}{\partial \mathbf{v}} = \left(\frac{\delta f_s}{\delta t}\right)_c$$

Contrast with Boltzmann's equation

<u>Boltzmann's H functional</u>

$$S(f) = -H(f) := -\int_{\Omega_x \times \mathbb{R}^3_v} f(x,v) \log f(x,v)\, dv\, dx$$

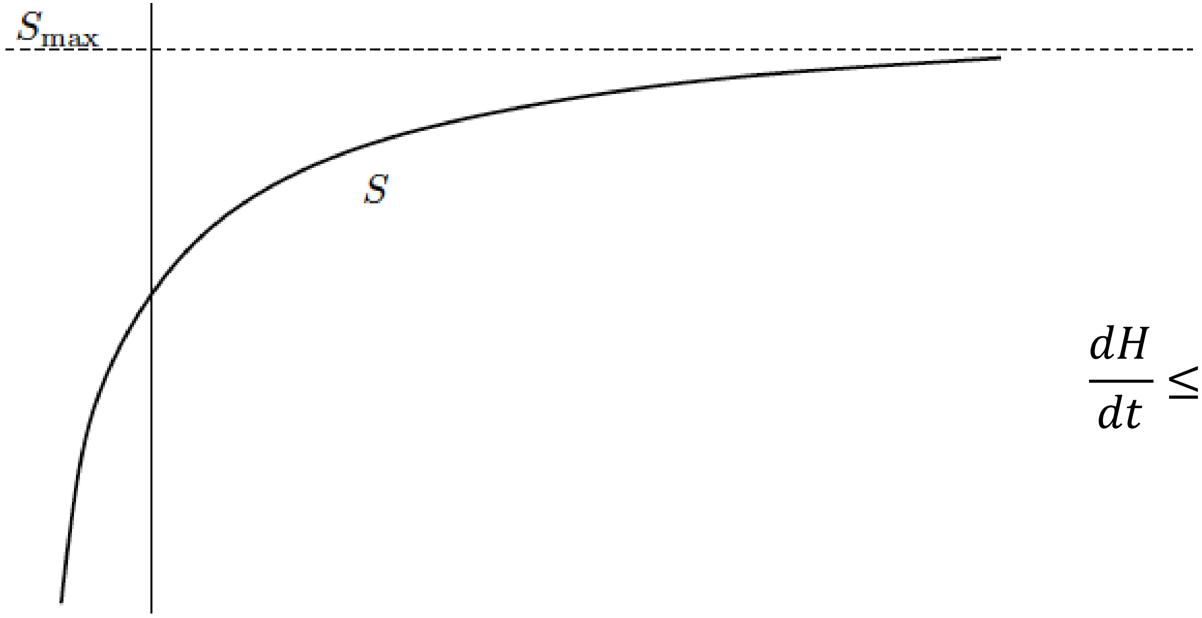
Boltzmann identifies S with the entropy of the gas

and proves that S can only increase in time

(strictly unless the gas is in a hydrodynamical state)

— an instance of the Second Law of Thermodynamics

Boltzmann's H-Theorem



$$\frac{dH}{dt} \leq 0, \qquad \frac{dS}{dt} \geq 0$$

If this is true, then f becomes Gaussian as $t \to \infty$!

Vlasov-Poisson equations: requirements of self-consistency in an electrostatic plasma

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \frac{\partial f_s}{\partial \mathbf{r}} - \frac{q_s}{m_s} \nabla \Phi \cdot \frac{\partial f_s}{\partial \mathbf{v}} = 0$$

$$\mathbf{E} = -\nabla \Phi$$

$$\nabla \cdot \mathbf{E} = -\nabla^2 \Phi = 4\pi\rho = 4\pi \sum_s q_s \int d\mathbf{v} f_s$$

Linear Plasma Waves

$$f_e(x, v, t) = f_{e0}(v) + f_{e1}(x, v, t)$$

$$f_{e0}(v) = n_{e0}\left(\frac{m_e}{2\pi k_B T_e}\right)^{1/2} \exp\left\{-\frac{m_e v^2}{2k_B T_e}\right\}$$

$$f_{e1} = \hat{f}_{e1} \exp[i(kx - \omega t)] \,.$$

Linearizing the Vlasov equation, and using the wave representation, we obtain

$$\frac{\partial f_{e1}}{\partial t} + v\frac{\partial f_{e1}}{\partial x} - \frac{e}{m_e} E_1 \frac{\partial f_{e0}}{\partial v} = 0$$

$$-i\omega \hat{f}_{e1} + ikv\hat{f}_{e1} - \frac{e}{m_e}\hat{E}_1 \frac{\partial f_{e0}}{\partial v} = 0\,,$$

Linear Plasma Waves

$$f_e(x, v, t) = f_{e0}(v) + f_{e1}(x, v, t)$$

$$f_{e0}(v) = n_{e0}\left(\frac{m_e}{2\pi k_B T_e}\right)^{1/2} \exp\left\{-\frac{m_e v^2}{2k_B T_e}\right\}$$

$$f_{e1} = \hat{f}_{e1} \exp[i(kx - \omega t)] .$$

$$\hat{f}_{e1} = i\frac{e}{m_e}\frac{\partial f_{e0}/\partial v}{\omega - kv}\hat{E}_1 .$$

The vanishing of the denominator $(\omega - kv)$ causes a singularity in the perturbed distribution function, which we will have to address carefully. The electrons with v=w/k are called *resonant particles*.

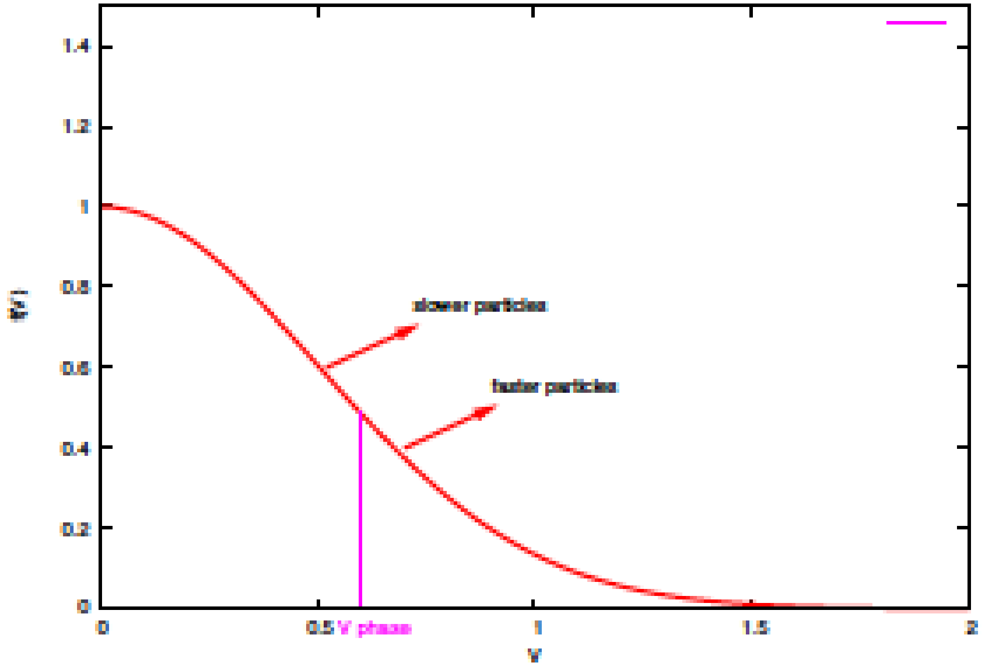
Linear Dispersion Relation

$$D(\mathbf{k}, \omega) = 1 - \sum_\alpha \frac{\omega_{p\alpha}^{2}}{k^2} \int_L \frac{\partial F_{\alpha 0}/\partial u}{u - \omega/|k|} \, du = 0$$

then gives for the dispersion equation for weakly damped electrostatic waves in a field-free plasma

$$1 - \sum_\alpha \frac{\omega_{p\alpha}^{2}}{k^2} \left(1 + i\omega_i \frac{\partial}{\partial \omega_r}\right) \oint \left[\frac{\partial F_{\alpha 0}(u)/\partial u}{u - \omega_r/|k|} \, du + \pi i \left[\frac{\partial F_{\alpha 0}(u)}{\partial u}\right]_{u = \omega_r/|k|} \right] = 0 \qquad (8.5.9)$$

Distribution function and Landau damping



$$\omega_l = \pm\omega_{pe}\left(1 + \frac{3}{2}k^2\lambda_D^2\right) + i\gamma_l(k)$$

$$\gamma_l(k) = -\left(\frac{\pi}{8}\right)^{1/2}\frac{\omega_{pe}}{k^3\lambda_D^3}\exp\left(-\frac{1}{2k^2\lambda_D^2} - \frac{3}{2}\right)$$

Non-Maxwellian Distributions

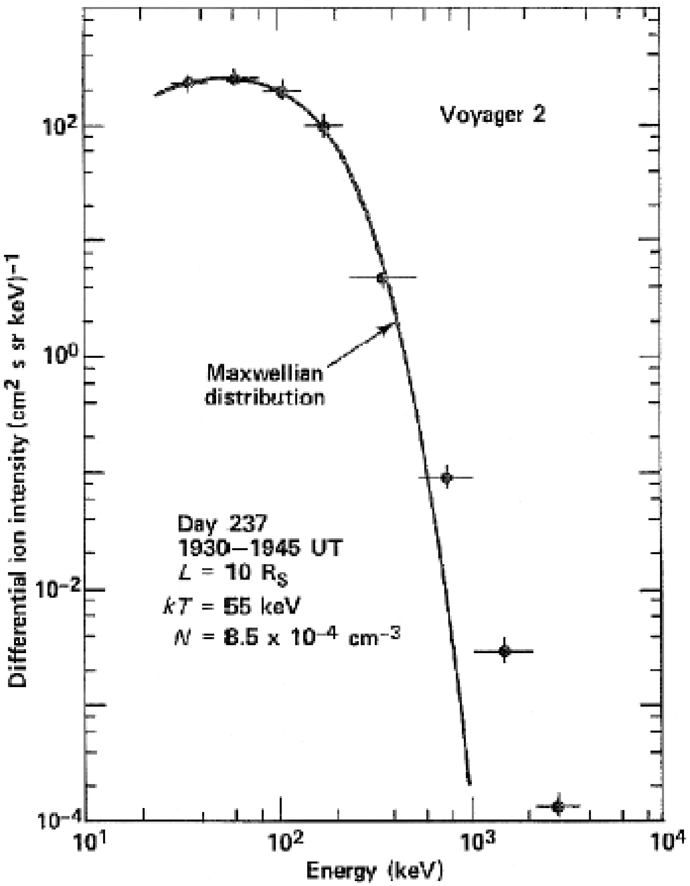


Figure 5. Typical ion energy spectrum in Saturn's magnetosphere measured by the LECP instrument on Voyager 2 at $10R_S$ (from (Krimigis, 1982)).

VOLUME 13, NUMBER 6 PHYSICAL REVIEW LETTERS 10 AUGUST 1964

COLLISIONLESS DAMPING OF ELECTROSTATIC PLASMA WAVES*

J. H. Malmberg and C. B. Wharton
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Landau Damping: The Measurement

Important key observation…

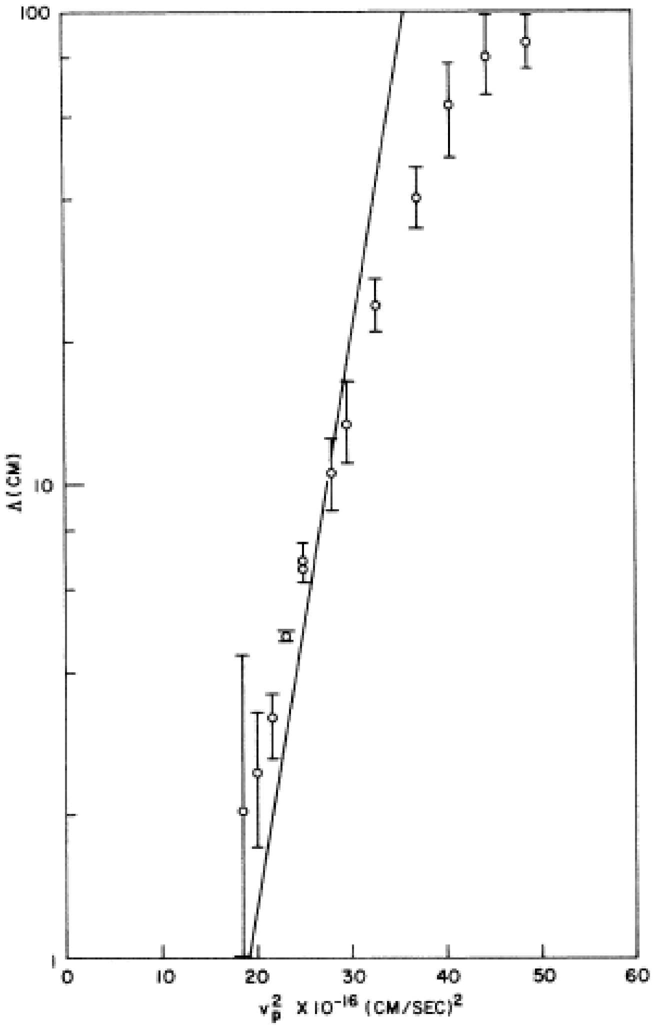

FIG. 3. Logarithm of damping length vs phase velocity squared. The solid curve is theory of Landau for a Maxwellian distribution with a temperature of 10.5 eV.

Quasilinear theory: application to scattering due to wave-particle interactions

- Consider electrostatic Vlasov equation

$$\frac{\partial f_s}{\partial t} + \mathbf{v}.\frac{\partial f_s}{\partial t} - \frac{q}{m}\nabla\Phi.\frac{\partial f_s}{\partial \mathbf{v}} = 0.$$

Split every dependent variable into a mean and a fluctuation

$$f_s = \langle f_s \rangle + f_{s1},\ \langle f_{s1} \rangle = 0$$

Quasilinear Diffusion

It follows after some algebra that the mean or average distribution function obeys a diffusion equation:

$$\frac{\partial}{\partial t}\langle f_s \rangle = \frac{\partial}{\partial \mathbf{v}} \cdot \left(\mathbf{D} \cdot \frac{\partial}{\partial \mathbf{v}} \langle f_s \rangle \right)$$

Here D is a diffusion tensor, dependent on wave fluctuations. These fluctuations can be a proxy for collisions as far as the average distribution function is concerned.

Fluid Models

A primary fluid model of focus in this summer school is Magnetohydrodynamics (MHD)

It treats the plasma as a single fluid, without distinguishing between electrons or protons, moving under the influence of self-consistent electric and magnetic fields.

It can be derived from kinetic theory by taking moments (integrating over velocity space), and making some drastic approximations.

Fluid equation of continuity

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_r f + \mathbf{a} \cdot \nabla_v f = 0$$

$$0 = \frac{\partial}{\partial t} \int f \,\mathrm{d}v + \frac{\partial}{\partial x} \int v\, f \,\mathrm{d}v + a[f]_{-\infty}^{\infty} = \frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nu)$$

Fluid momentum equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_r f + \mathbf{a} \cdot \nabla_v f = 0$$

$$\begin{aligned}
0 &= \frac{\partial}{\partial t} \int m v f \, \mathrm{d}v + \frac{\partial}{\partial x} \int \overset{\times m}{v^2 f \, \mathrm{d}v} + a \int v \frac{\partial f}{\partial v} \, \mathrm{d}v \quad \times m \\
&= \frac{\partial}{\partial t} \int m v f \, \mathrm{d}v + \frac{\partial}{\partial x} \left[\int m(v-u)^2 f \, \mathrm{d}v + n m u^2 \right] \\
&\quad + a \left([v f]_{-\infty}^{\infty} - \int f \frac{\mathrm{d}v}{\mathrm{d}v} \, \mathrm{d}v \right) \times m \\
&= \frac{\partial}{\partial t}(n m u) + \frac{\partial p}{\partial x} + u \frac{\partial}{\partial t}(n m u) + (n m u) \frac{\partial u}{\partial x} - n m a \\
&= n m \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right) + \frac{\partial p}{\partial x} - n m a \, ,
\end{aligned}$$

$$p = \int m(v-u)^2 f \, \mathrm{d}v$$