TURBULENCE IN FLUIDS AND SPACE PLASMAS

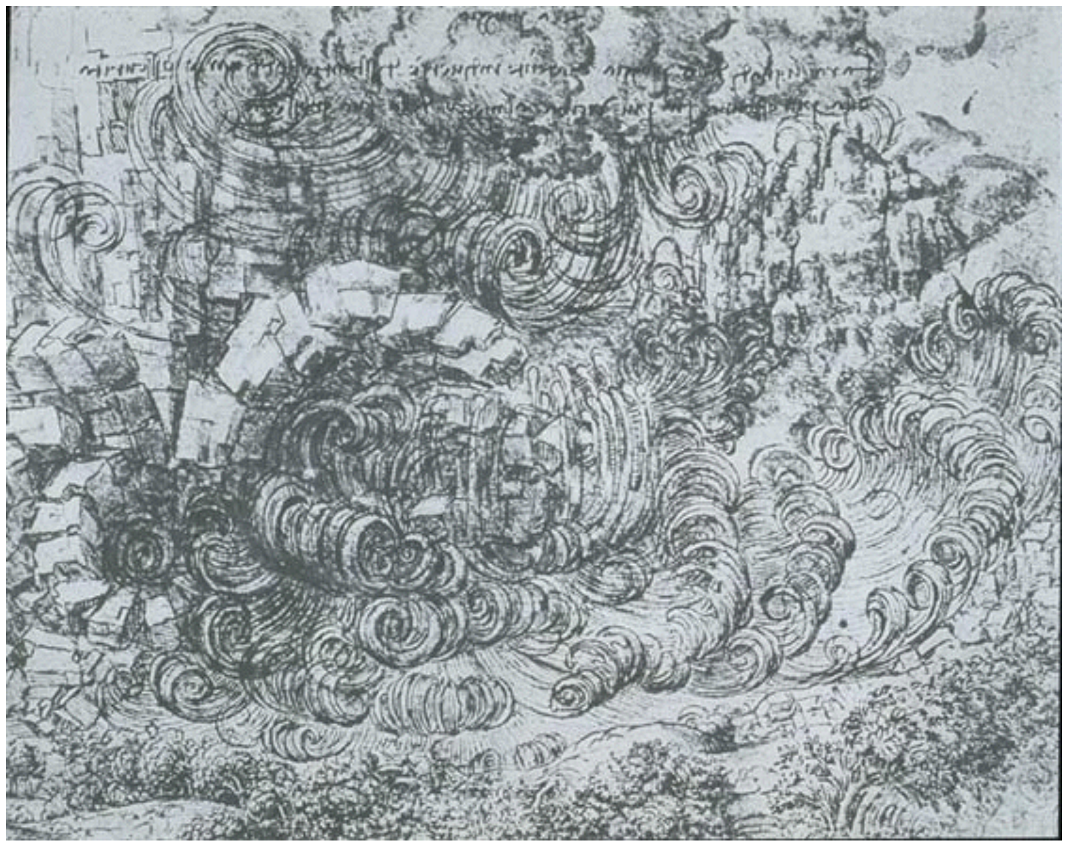
Amitava Bhattacharjee

Princeton Plasma Physics Laboratory, Princeton University

What is Turbulence?

- Webster's 1913 Dictionary: "*The quality or state of being turbulent; a disturbed state; tumult; disorder, agitation.*"

- Alexandre J. Chorin, Lectures on Turbulence Theory (Publish or Perish, Inc., Boston 1975): "*The distinguishing feature of turbulent flow is that its velocity field appears to be random and varies unpredictably. The flow does, however, satisfy a set of……equations, which are not random. This contrast is the source of much of what is interesting in turbulence theory.*"



Leonardo da Vinci (1500)

Navier-Stokes Equation: Fundamental Equation for Fluid Turbulence

$$\frac{\partial v}{\partial t} + v \cdot \nabla v = -\nabla p + R^{-1}\nabla^2 v + F$$

$$\nabla \cdot v = 0$$

R (= LV/viscosity) is called the Reynolds number.

What is the effect of viscosity?

- Experimental fact, not at all obvious: *The velocity of a viscous fluid is exactly zero at the surface of a solid.*
- *Example*: the blade of a fan collects a thin layer of fine dust that persists even when the fan churns up air all around it.
- So near a wall the velocity of a fluid can change rapidly from large values to zero. The narrow region where the velocity can change rapidly is called a "boundary layer." In the boundary layer, the effect of viscosity is very important.

Richard Feynman, Lectures on Physics, Volume II

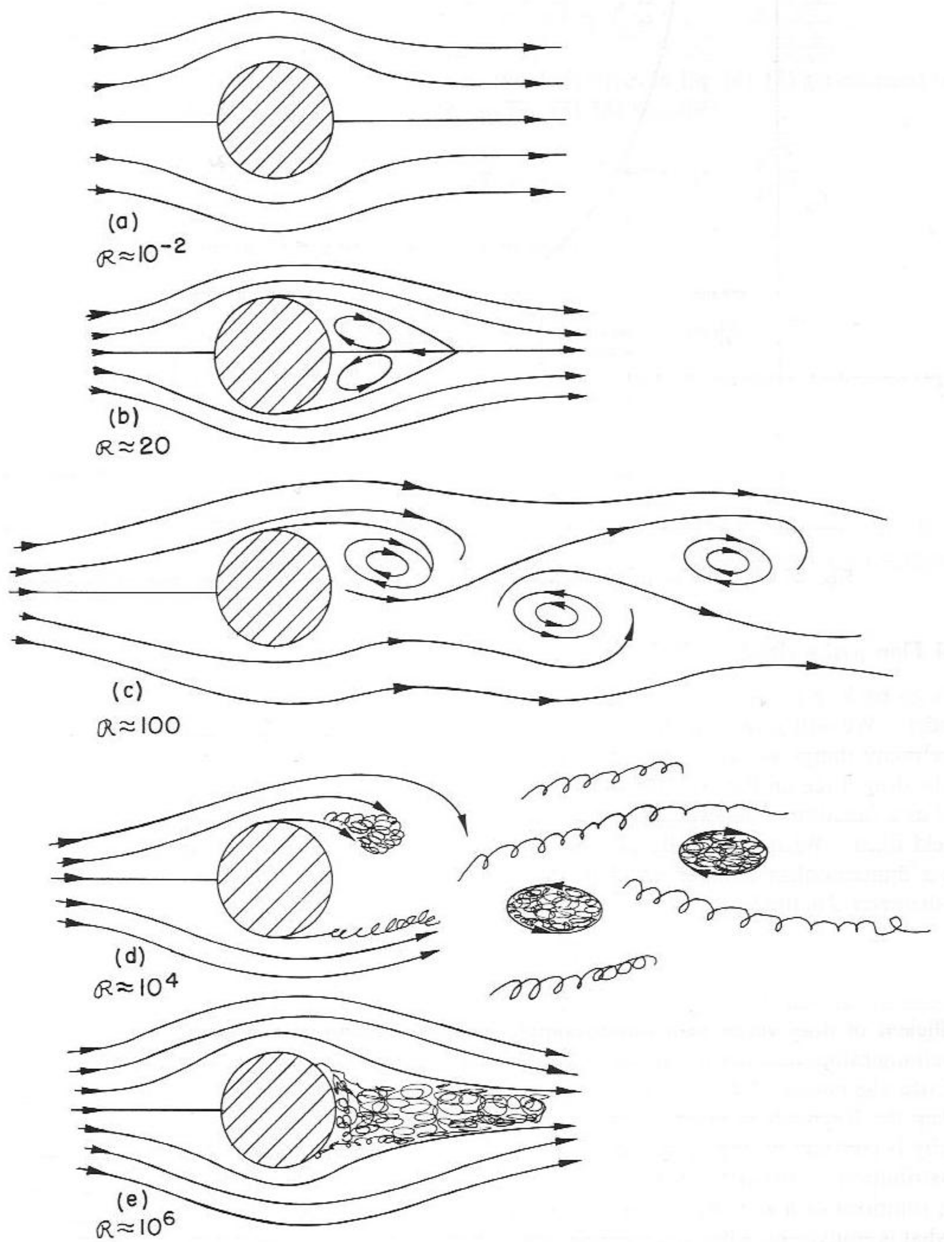


Fig. 41–6. Flow past a cylinder for various Reynolds numbers.

Flow past a circular cylinder

- When R<<1, the lines of v are curved, open lines and constant in time.
- When R is in the range 10-30, a pair of vortex structures develop behind the cylinder. The velocity, though more complex in space, is a constant in time.
- When R increases above 40, the character of the flow changes. The vortices behind the cylinder get so long that they just break off and travel downstream, forming a so-called "vortex street." The velocity is no longer constant in time everywhere, but varies in a regular cyclic fashion.

Flow past a circular cylinder: how does turbulence come about?

- When R is increased further (that is, viscosity is lower), the vortex structures get trapped into narrower regions (boundary layers). Within these regions the flow is rapidly varying in time.
- When R increases even further, the turbulent works its way in to the point that the velocity flow lines leave the cylinder. We have a "turbulent boundary layer", in which the velocity is very irregular and noisy. This velocity is a very sensitive function of the initial conditions----small changes in initial conditions can produce very large changes in the final results.
- Velocity thus appears to be "random" or "chaotic", although the underlying equations are not random.

Turbulence: Spatial Characteristics

- Turbulence couples large scales and small scales.
- The process of development of turbulence often starts out as large-scale motion by the excitation of waves of long wavelength that quickly produces waves of small wavelength by a domino effect.

A. N. Kolmogorov (1941): Universal Inertial Range Spectrum for Turbulent Flows

$$E(k) \propto k^{-5/3}$$

Inertial range is the range of wavenumbers between the forcing scale at small wavenumber (large wavelength) and the dissipation scale at large wavenumber (small wavelength)

Hydrodynamic turbulence

Kolmogorov (1941): energy spectrum by dimensional analysis

Assumptions: ♦ isotropy

♦ local interaction in k-space

(energy moves from one k-shell to the next)

Energy cascade rate: $\varepsilon(k) \propto k^{\alpha} \underbrace{E(k)}^{\beta}$ = constant

Energy spectrum: $\int E(k)dk =$ total energy

$\Rightarrow \alpha = 5/2, \quad \beta = 3/2$

Kolmogorov spectrum: $E(k) \sim C_{\mathrm{K}} \varepsilon^{2/3} k^{-5/3}$

Kolmogorov constant: $C_{\mathrm{K}} \sim 1.4 - 2$

E(*k*) in inertial subrange: Dimensional analysis

$$[\mathrm{E}] = m^2 s^{-2} \ ; \quad [\varepsilon] = m^2 s^{-3} \ ; \quad [k] = m^{-1} \ ;$$

$$[E(k)] = [\mathrm{E}]/[k] = m^3 s^{-2}$$

$$Dimensional\ analysis: \quad [\varepsilon^{2/3} k^{-5/3}] = m^3 s^{-2}$$

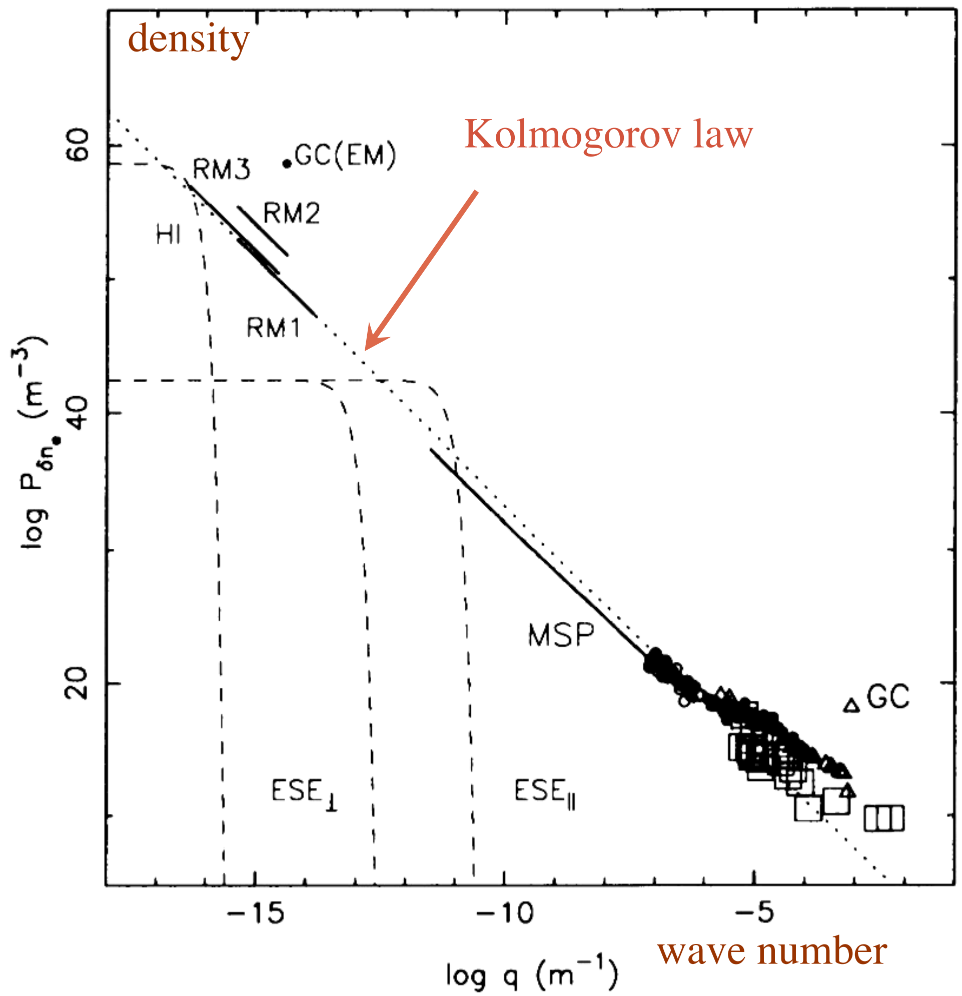
$$\Rightarrow \ E(k) \ \propto \ \varepsilon^{2/3} k^{-5/3}$$

$$\Rightarrow \ E(k) \ = \ C\,\varepsilon^{2/3} \kappa^{-5/3}$$

The last equation describes the famous Kolmogorov –5/3 spectrum. C is the universal Kolmogorov constant, which experimentally was determined to be C = 1.5.

Interstellar turbulence

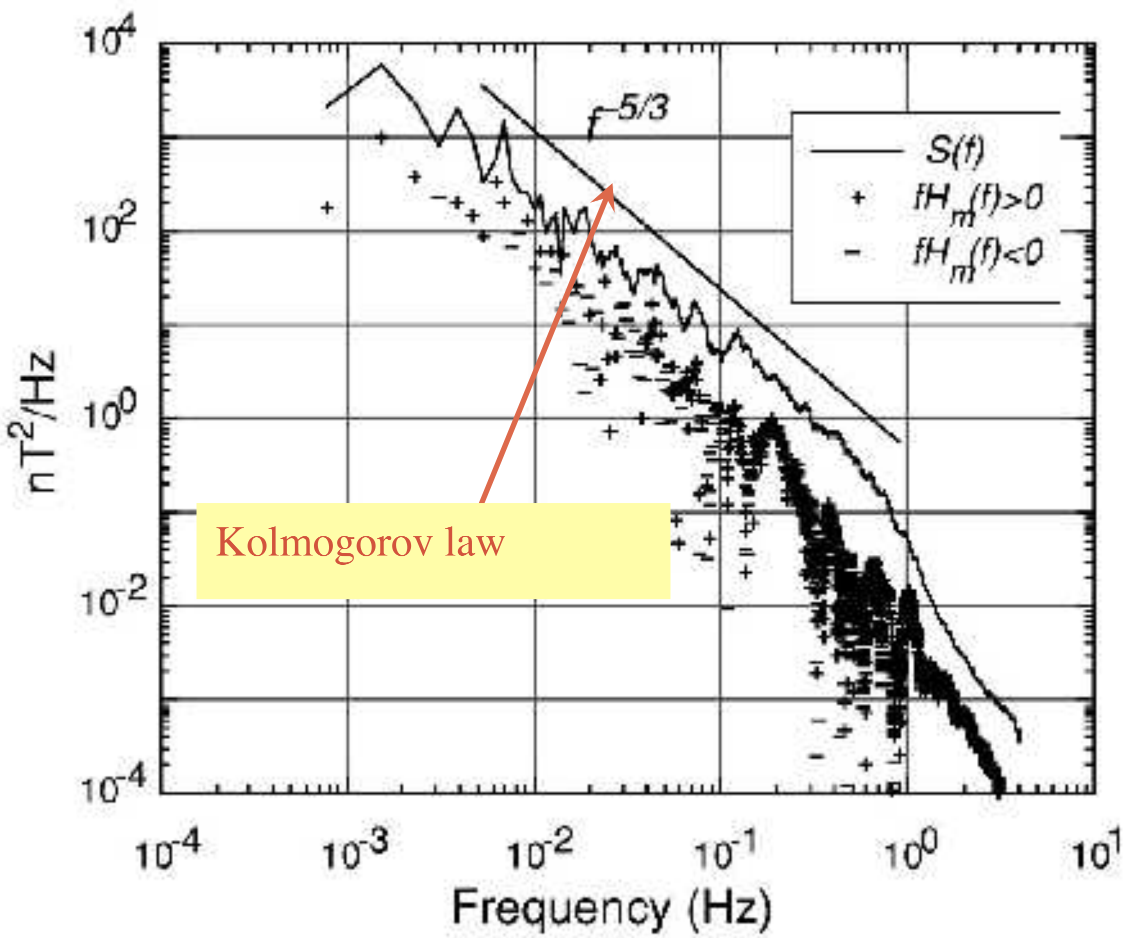
Observation:
power law relation between electron density spectrum and spatial scales



From Cordes (1999)

Solar wind turbulence

Observation: power law in magnetic energy spectrum

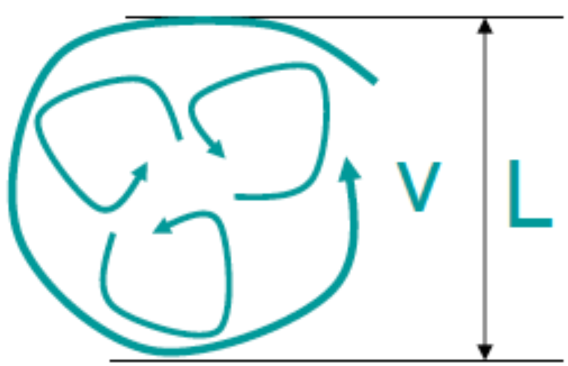


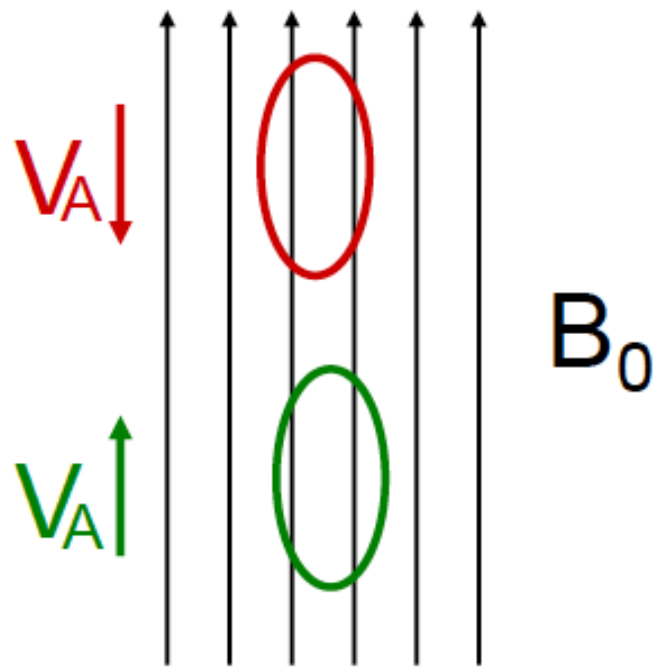
From Goldstein & Roberts (1999)

Nature of Magnetohydrodynamic (MHD) turbulence

HD turbulence:
interaction of eddies

MHD turbulence:
interaction of wave packets
moving with Alfven velocities





$$\mathbf{v}_A = \mathbf{B}_0/\sqrt{4\pi\rho_0}$$

S. Boldyrev

Magnetohydrodynamic (MHD) equations

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho_0}\nabla p + \frac{1}{4\pi\rho_0}(\mathbf{B} \cdot \nabla)\mathbf{B} + \nu\nabla^2\mathbf{v}$$
$$\partial_t \mathbf{B} = \nabla \times [\mathbf{v} \times \mathbf{B}] + \eta\nabla^2\mathbf{B}$$

Separate the uniform magnetic field: $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$

Introduce the Elsasser variables: $\mathbf{z}^{\pm} = \mathbf{v} \pm \frac{1}{\sqrt{4\pi\rho_0}}\mathbf{b}$

Then the equations take a symmetric form:

$$\partial_t \mathbf{z}^+ - (\mathbf{v}_A \cdot \nabla)\mathbf{z}^+ + (\mathbf{z}^- \cdot \nabla)\mathbf{z}^+ = -\nabla P$$
$$\partial_t \mathbf{z}^- + (\mathbf{v}_A \cdot \nabla)\mathbf{z}^- + (\mathbf{z}^+ \cdot \nabla)\mathbf{z}^- = -\nabla P$$

With the Alfven velocity $\mathbf{v}_A = \mathbf{B}_0/\sqrt{4\pi\rho_0}$

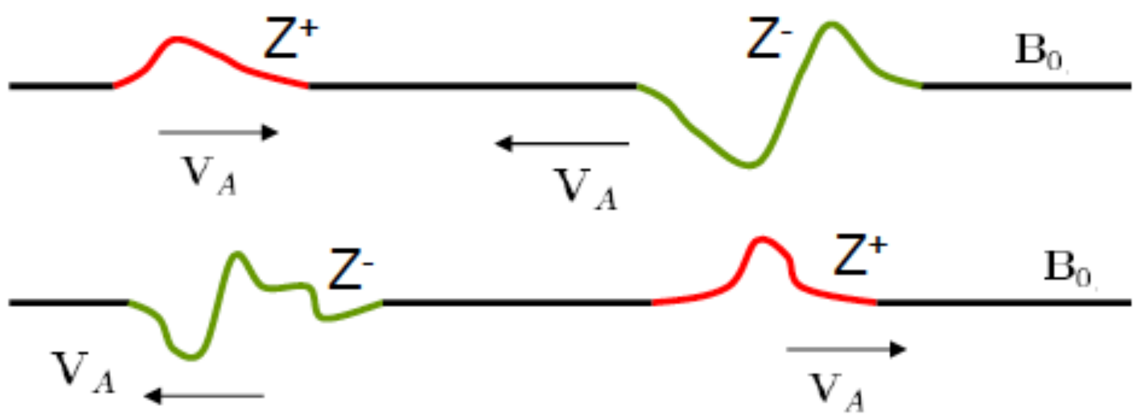
The uniform magnetic field mediates small-scale turbulence

MHD turbulence: Alfvenic cascade

$$\partial \mathbf{z}^{\pm} \mp (\mathbf{v}_A \cdot \nabla)\mathbf{z}^{\pm} + (\mathbf{z}^{\mp} \cdot \nabla)\mathbf{z}^{\pm} = -\nabla P + \frac{1}{Re}\nabla^2 \mathbf{z}^{\pm} + \mathbf{f}^{\pm}$$

Ideal system conserves the Elsasser energies

$$E^+ = \int (\mathbf{z}^+)^2 \, d^3x \qquad E = \frac{1}{2}\int (v^2 + b^2) d^3x$$
$$\equiv$$
$$E^- = \int (\mathbf{z}^-)^2 \, d^3x \qquad H^C = \int (\mathbf{v} \cdot \mathbf{b}) d^3x$$



$E^+ \sim E^-$: balanced case. $\qquad$ $E^+ \gg E^-$: imbalanced case

$$H^C = \int (\mathbf{v} \cdot \mathbf{b}) d^3x = \frac{1}{4}(E^+ - E^-) \neq 0$$

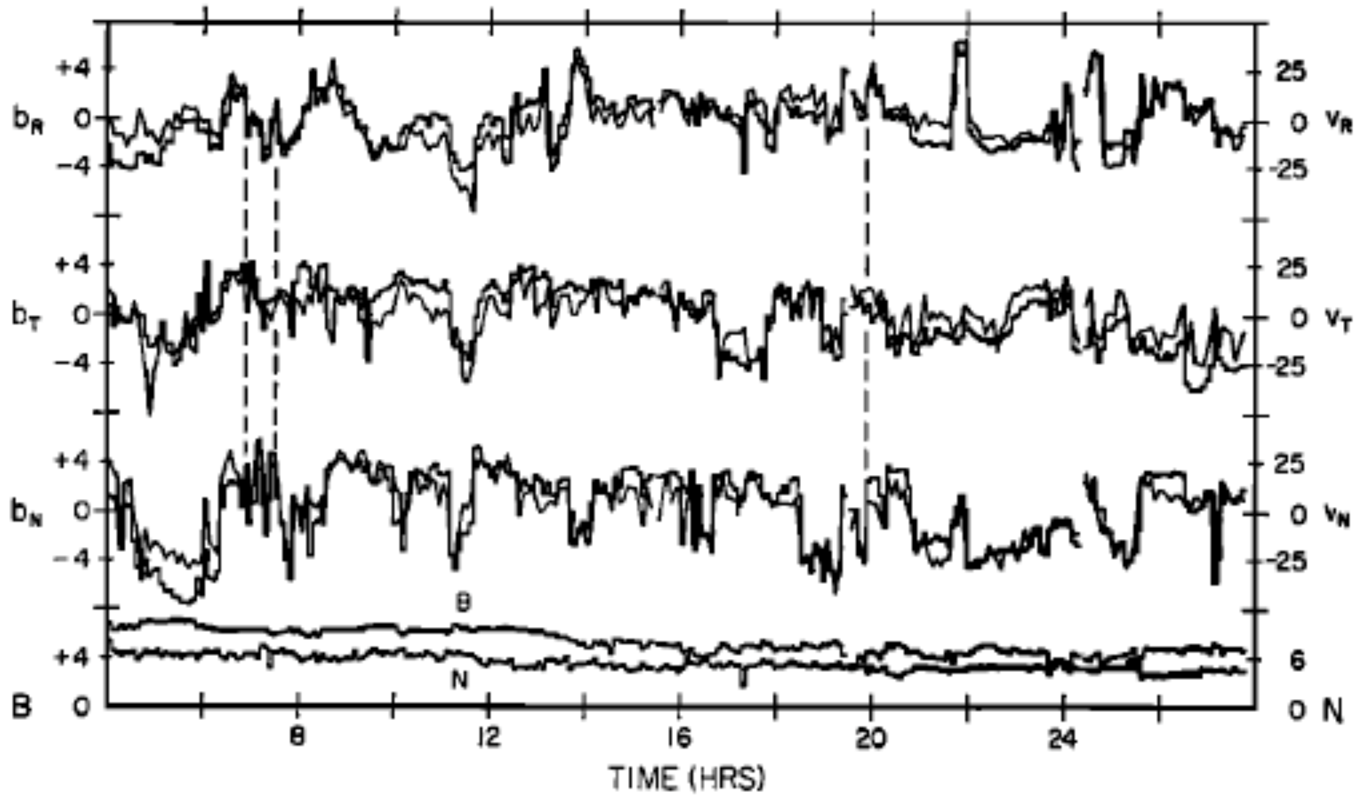


Fig. 1. Twenty-four hours of magnetic field and plasma data demonstrating the presence of nearly pure Alfvén waves. The upper six curves are 5.04-min bulk velocity components in km/sec (diagonal lines) and magnetic field components averaged over the plasma probe sampling period, in gammas (horizontal and vertical lines). The lower two curves are magnetic field strength and proton number density.

MHD turbulence

Additional dependence on the Alfvén speed V_A

Energy cascade rate: $\varepsilon(k) \propto k^{\alpha} E(k)^{\beta} V_A^{\gamma}$ = constant

$$\Rightarrow \alpha = (5-\gamma)/2, \quad \beta = (3-\gamma)/2$$

Spectral index: $\gamma = \alpha/\beta = (5-\gamma)/(3-\gamma)$

$\gamma = 0$: Kolmogorov spectrum $\qquad E(k) \sim C_{\mathrm{K}} \varepsilon^{2/3} k^{-5/3}$

$\gamma = -1$: IK spectrum $\qquad E(k) \sim C_{IK} \varepsilon^{1/2} V_A^{1/2} k^{-3/2}$

Iroshnikov (1963), Kraichnan (1965) $\qquad C_{IK} \sim 1.8 - 2.2$

$$\Longrightarrow \quad z_\ell(t+\tau_A) \sim z_\ell(t) + \tau_A \frac{\partial z_\ell}{\partial t} \sim z_\ell(t) + \tau_A \frac{z_\ell^2}{\ell} \qquad z^+ \sim z^- \sim z$$

Therefore, the deformation of the wave-packet after one collision is :

$$\Delta_1 z_\ell \sim \tau_A \frac{z_\ell^2}{\ell} \, .$$

This deformation will grow with time and for N stochastic collisions effect can be evaluated in the same manner as a random walk :

$$\sum_{i=1}^{N} \Delta_i z_\ell \sim \tau_A \frac{z_\ell^2}{\ell} \sqrt{\frac{t}{\tau_A}} \, .$$

S. Galtier

Cumulative distorsion of order one defines τ_{tr} : $z_\ell \sim \sum_1^N \Delta_i z_\ell \sim \tau_A \frac{z_\ell^2}{\ell} \sqrt{\frac{\tau_{tr}}{\tau_A}}$

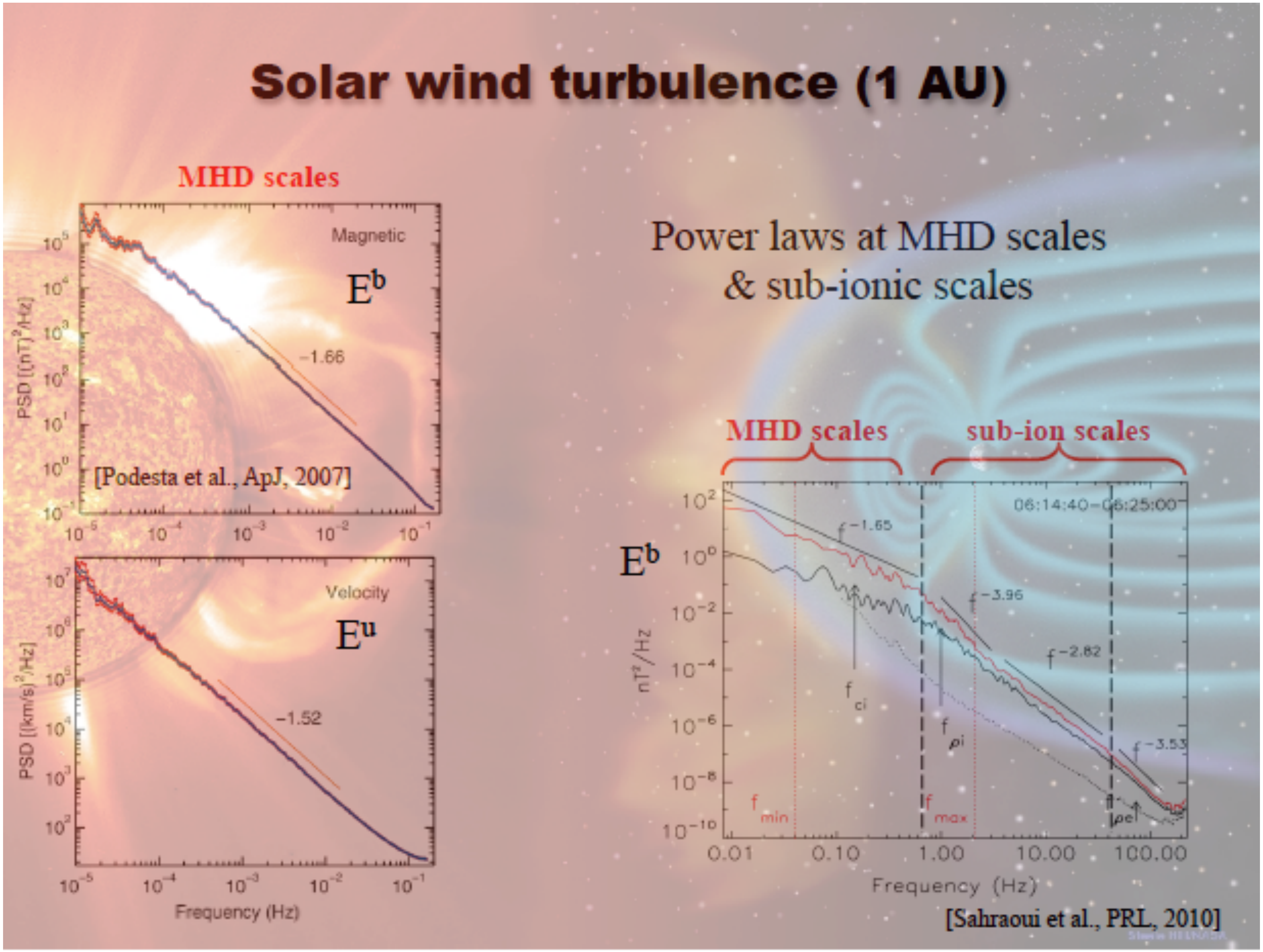
$$\Rightarrow \quad \tau_{tr} \sim \frac{1}{\tau_A} \frac{\ell^2}{z_\ell^2} \sim \frac{\tau_{eddy}^2}{\tau_A} \qquad \tau_{eddy} \sim \ell / u_\ell$$

$$\Rightarrow \quad \varepsilon \sim \frac{z_\ell^2}{\tau_{eddy}^2/\tau_A} \sim \frac{z_\ell^4}{\ell b_0} \sim \frac{E^2(k)k^3}{b_0} \quad \Rightarrow$$

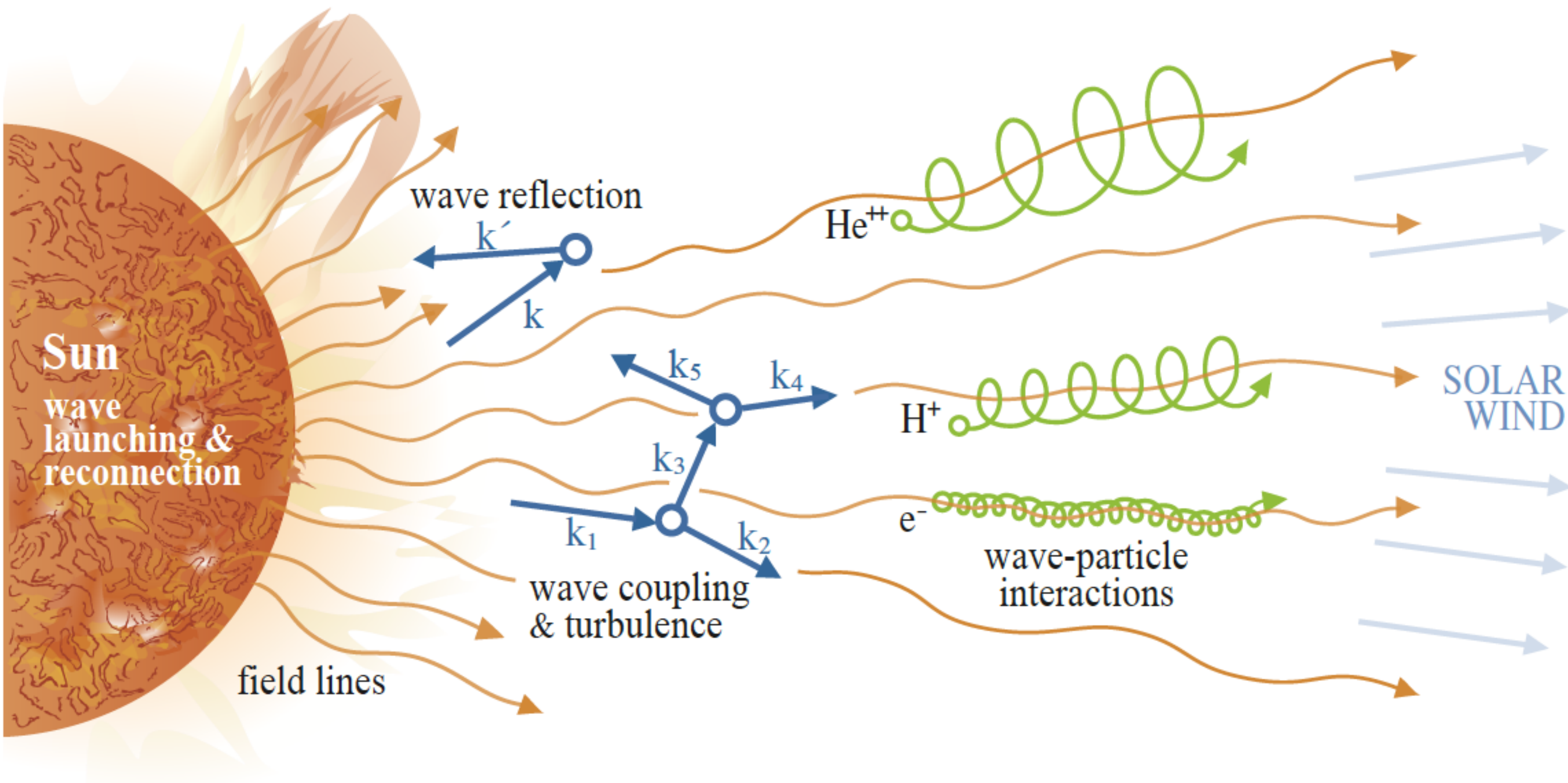
IK SPECTRUM

$$\boxed{E^z(k) = C_{IK}\sqrt{\varepsilon b_0}\, k^{-3/2}}$$

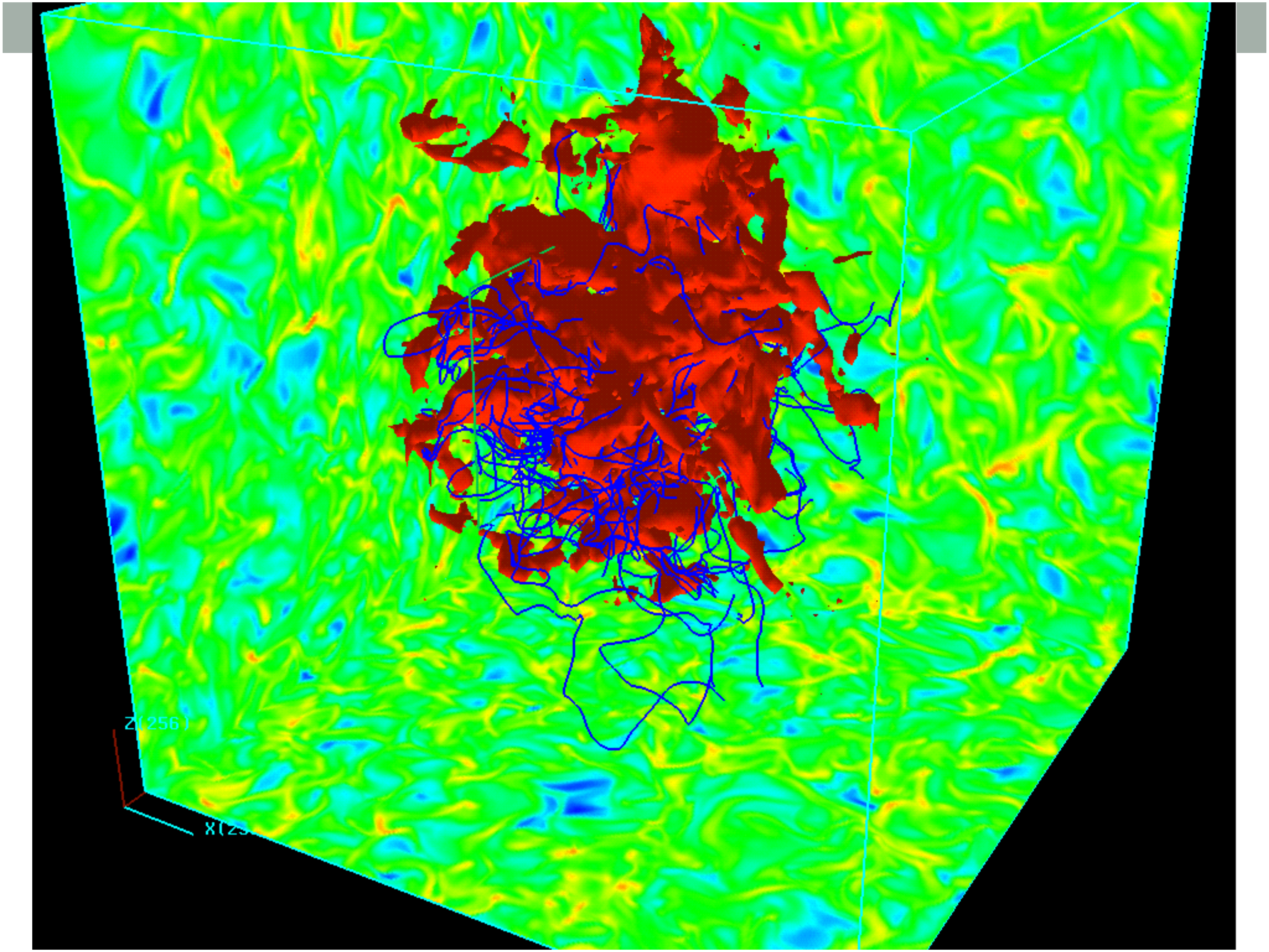
Iroshnikov 1964, Kraichnan 1965

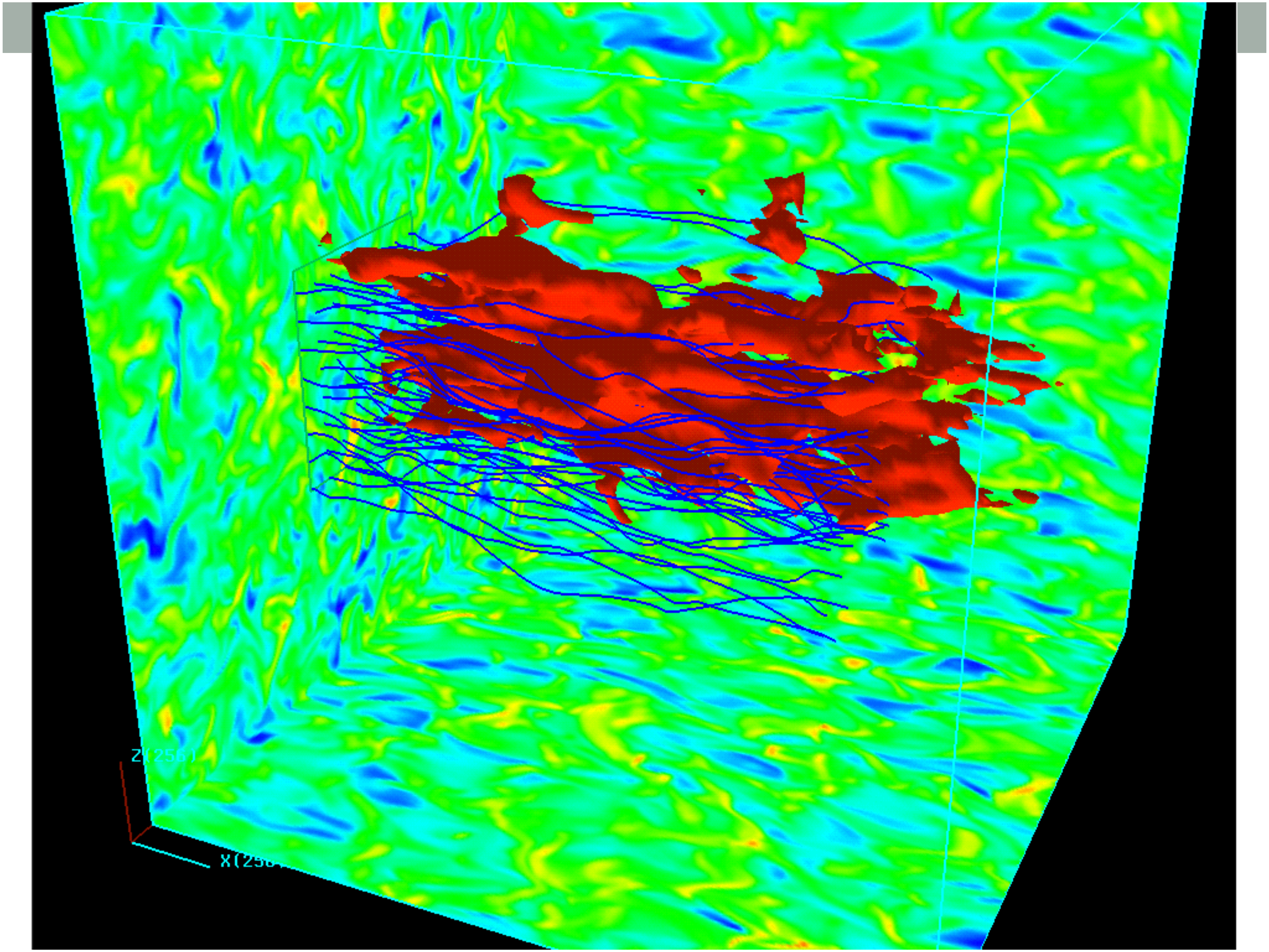


Solar wind turbulence (1 AU)
MHD scales
Magnetic
E^b
-1.66
[Podesta et al., ApJ, 2007]
Velocity
E^u
-1.52
Frequency (Hz)
Power laws at MHD scales
& sub-ionic scales
MHD scales
sub-ion scales
E^b
06:14:40-06:25:00
f^-1.65
f^-3.96
f^-2.82
f^-3.53
f_ci
f_ρi
f_min
f_max
nT²/Hz
Frequency (Hz)
[Sahraoui et al., PRL, 2010]



M. Lee/B. Chandran/K. Donahue


Z(256)
X(25


Z(256)

Strength of interaction in MHD turbulence

$$\partial \mathbf{z}^{\pm} \mp \underbrace{(\mathbf{v}_A \cdot \nabla)\mathbf{z}^{\pm}}_{(k_{\|} v_A) z^{\pm}} + \underbrace{(\mathbf{z}^{\mp} \cdot \nabla)\mathbf{z}^{\pm}}_{(k_{\perp} z^{\mp}) z^{\pm}} = -\nabla P + \frac{1}{Re}\nabla^2 \mathbf{z}^{\pm} + \mathbf{f}^{\pm}$$

When $k_{\|} v_A \gg k_{\perp} z^{\mp}$ turbulence is weak

When $k_{\|} v_A \sim k_{\perp} z^{\mp}$ turbulence is strong

Wave MHD turbulence: Phenomenology

Three-wave interaction of shear-Alfven waves

$$\omega(k) = |k_z| v_A$$

$$\begin{cases} \omega(k) = \omega(k_1) + \omega(k_2) \\ \mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2 \end{cases}$$

Only counter-propagating waves interact, therefore, k_{1z} and k_{2z} should have opposite signs.

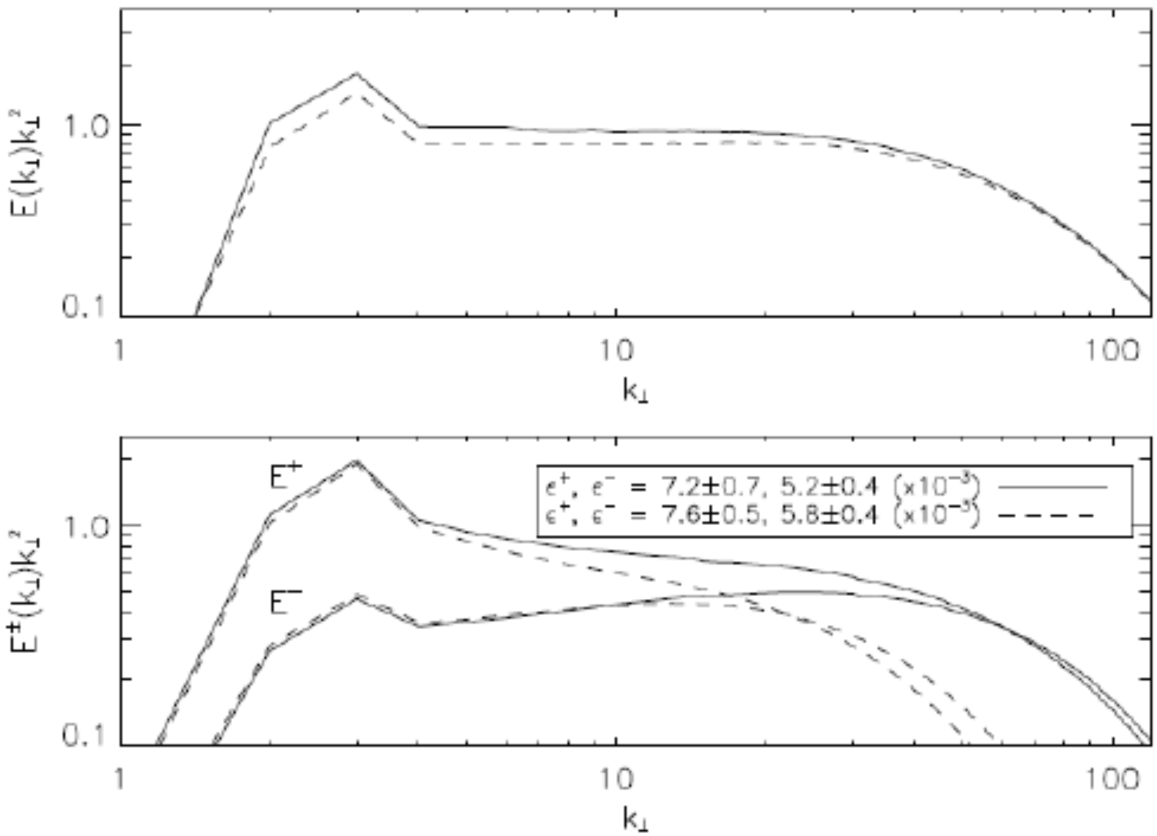
$$\Downarrow$$

Either $k_{1z} = 0$ or $k_{2z} = 0$

Wave interactions change $k_\perp$ but not k_z

At large $k_\perp$: $E(k_z, k_\perp) \propto g(k_z) k_\perp^{-\beta}$

Imbalanced weak MHD turbulence: Numerical results



Balanced

Imbalanced

[Boldyrev & Perez (2009)]

Phenomenological theories of incompressible strong MHD turbulence

- Anisotropic **Goldreich-Sridhar** theory (1995)

$$E(k_\perp) \propto k_\perp^{-5/3}$$

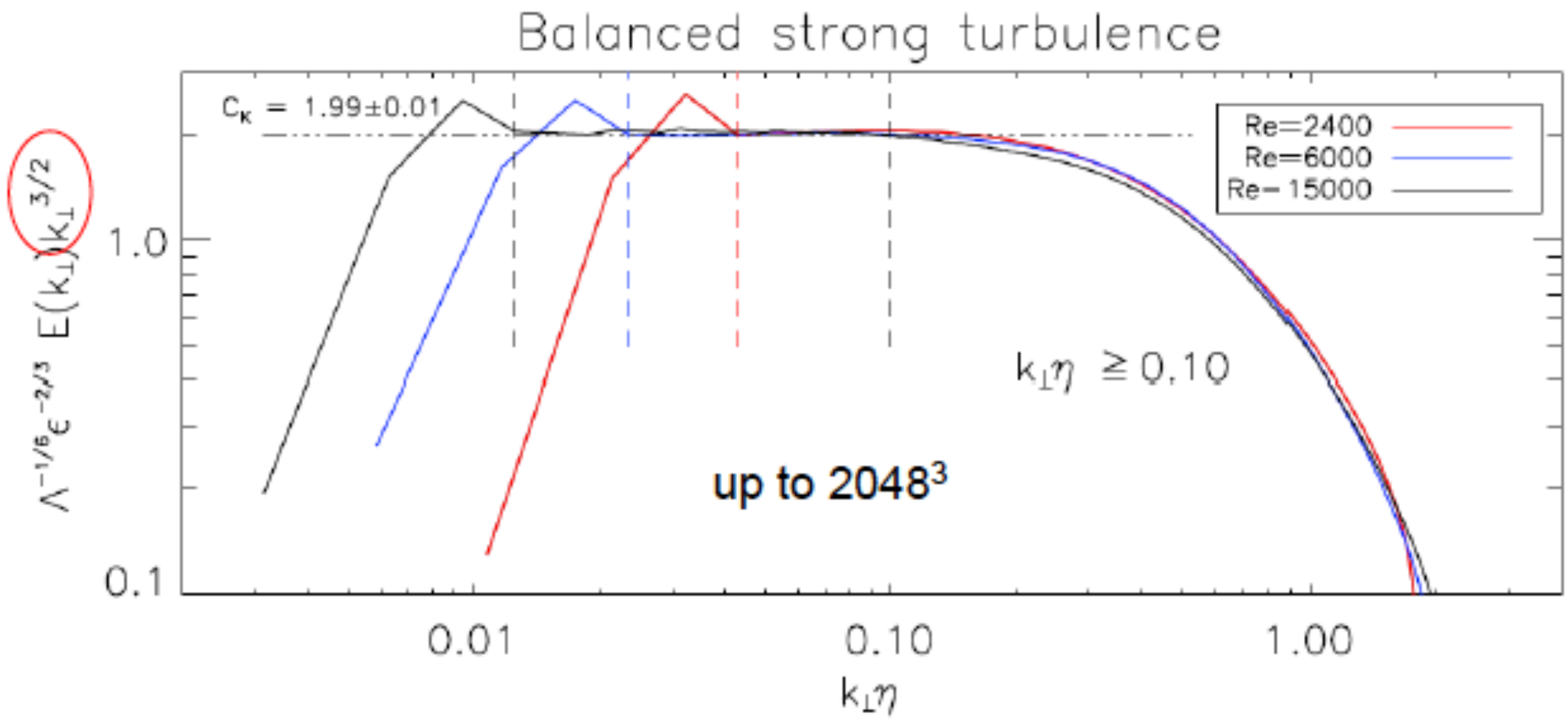
- Advances in 3D **Numerical simulations** (1996-2005) have shown

$$E(k_\perp) \propto k_\perp^{-3/2}$$ Magnetized plasmas

- **Boldyrev** (2006): Need to modify GS95 to explain energy spectrum seen in simulations.

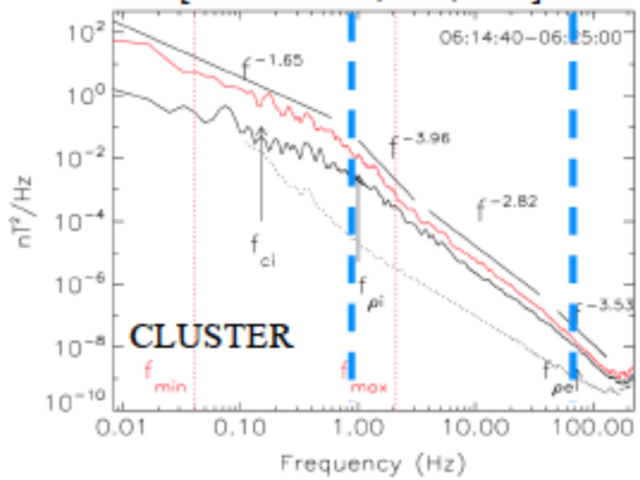
$$E(k_\perp) \propto k_\perp^{-3/2}$$

Spectrum of strong MHD turbulence: balanced case



Solar wind turbulence at sub-ion scales

[Sahraoui et al., PRL, 2010]



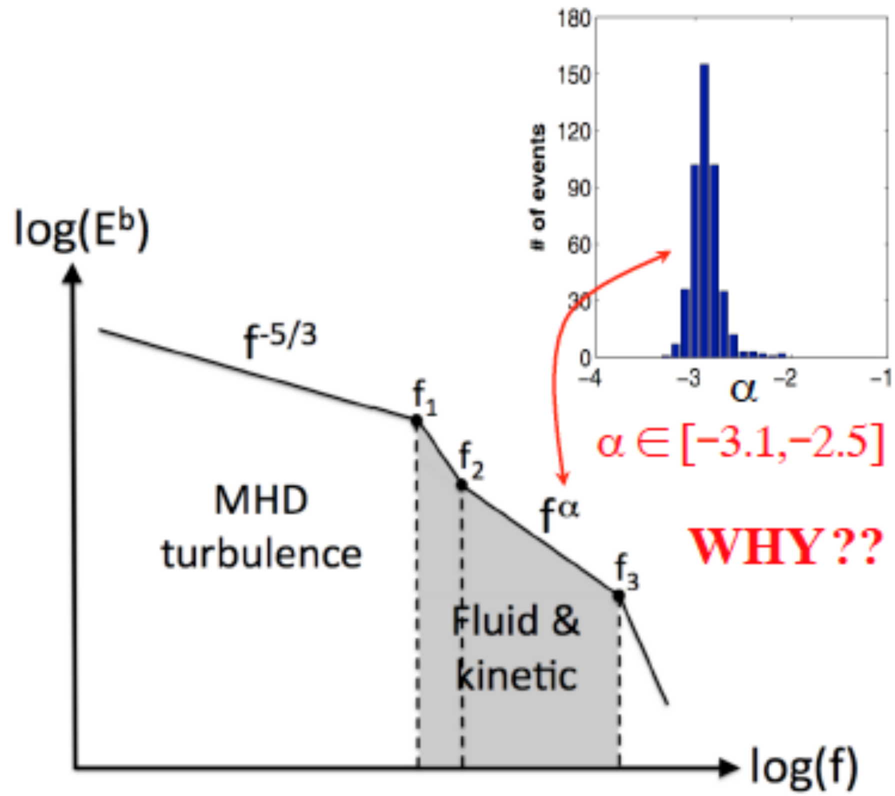
$T_i/T_e \sim 1$

$\beta_i \sim \beta_e \sim 1$

$d_i \sim 100\,\mathrm{km}$

$f_{ci} \sim 0.1\,\mathrm{Hz}$

[Sahraoui et al., ApJ, 2013]



What is the origin of the scaling laws of the magnetic fluctuation spectra observed beyond f_2 ?