DYNAMO THEORY: BASICS

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Magnetic fields in the Universe

- Earth
  - Magnetic field present for $\sim 3.5 \cdot 10^9$ years, much longer than Ohmic decay time ($\sim 10^4$ years)
  - Strong variability on shorter time scales (10^3 years)
- Planets: Mercury, Jupiter, Saturn, Uranus, Neptune have large scale fields
- Sun
  - Magnetic fields from smallest observable scales to size of sun
  - 11 year cycle of large scale field
  - Ohmic decay time $\sim 10^9$ years (in absence of turbulence)
- Other stars
  - Stars with outer convection zone: similar to sun
  - Stars with outer radiation zone: most likely primordial fields
- Galaxies
  - Field structure coupled to observed matter distribution (e.g. spirals)
  - Is it primordial?

Why is the Universe magnetized?

D. Longcope lecture: Dynamo has 3 fundamental features:

- Electrically conducting fluid
- Fluid must have complex motions
- Motions must be vigorous enough (as measured by the Magnetic Reynolds number Rm = velocity x size/ resistivity)

Toy example: Homopolar Dynamo

References for this lecture:

Heliophysics, Vol. 1, Chapter 3: M. Rempel

Heliophysics, Vol. 3, Chapter 6: P. Charbonneau

Heliophysics, Vol. 4, Chapter 6: S. Stanley

MHD equations

The full set of MHD equations combines the induction equation with the Navier-Stokes equations including the Lorentz-force:

$$
\begin{aligned}
\frac{\partial \varrho}{\partial t} &= -\nabla \cdot (\varrho \mathbf{v}) \\
\varrho \frac{\partial \mathbf{v}}{\partial t} &= -\varrho(\mathbf{v} \cdot \nabla)\mathbf{v} - \nabla p + \varrho \mathbf{g} + \frac{1}{\mu_0}(\nabla \times \mathrm{B}) \times \mathrm{B} + \nabla \cdot \tau \\
\varrho \frac{\partial e}{\partial t} &= -\varrho(\mathbf{v} \cdot \nabla)e - p\nabla \cdot \mathbf{v} + \nabla \cdot (\kappa \nabla T) + Q_\nu + Q_\eta \\
\frac{\partial \mathrm{B}}{\partial t} &= \nabla \times (\mathbf{v} \times \mathrm{B} - \eta \nabla \times \mathrm{B})
\end{aligned}
$$

Assumptions:

- Validity of continuum approximation (enough particles to define averages)
- Non-relativistic motions, low frequencies

MHD equations

Viscous stress tensor τ

$$\begin{aligned}\Lambda_{ik} &= \frac{1}{2}\left(\frac{\partial v_i}{\partial x_k}+\frac{\partial v_k}{\partial x_i}\right)\\ \tau_{ik} &= 2\varrho\nu\left(\Lambda_{ik}-\frac{1}{3}\delta_{ik}\boldsymbol{\nabla}\cdot\mathbf{v}\right)\\ Q_\nu &= \tau_{ik}\Lambda_{ik}\ ,\end{aligned}$$

Ohmic dissipation Q_η

$$Q_\eta = \frac{\eta}{\mu_0}(\boldsymbol{\nabla}\times\mathbf{B})^2\ .$$

Equation of state

$$p = \frac{\varrho\, e}{\gamma-1}\ .$$

ν, η and κ: viscosity, magnetic diffusivity and thermal conductivity

μ_0 denotes the permeability of vacuum

Induction equation

Using Ampere's law $\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$ yields for the electric field in the laboratory frame

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B} + \frac{1}{\mu_0 \sigma} \nabla \times \mathbf{B}$$

leading to the induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} = \nabla \times (\mathbf{v} \times \mathbf{B} - \eta \nabla \times \mathbf{B})$$

with the magnetic diffusivity

$$\eta = \frac{1}{\mu_0 \sigma} .$$

Advection, diffusion, magnetic Reynolds number

L: typical length scale U: typical velocity scale L/U: time unit

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left(\mathbf{v} \times \mathbf{B} - \frac{1}{R_m} \nabla \times \mathbf{B} \right)$$

with the magnetic Reynolds number

$$R_m = \frac{U\,L}{\eta} \ .$$

$R_m \ll 1$: diffusion dominated regime

$$\frac{\partial \mathbf{B}}{\partial t} = \eta \Delta \mathbf{B} \ .$$

Only decaying solutions with decay (diffusion) time scale

$$\tau_{\mathrm{d}} \sim \frac{L^2}{\eta}$$

Advection, diffusion, magnetic Reynolds number

$R_m \gg 1$ advection dominated regime (ideal MHD)

$$\frac{\partial \mathbf{B}}{\partial t} = \boldsymbol{\nabla} \times (\mathbf{v} \times \mathbf{B})$$

Equivalent expression

$$\frac{\partial \mathbf{B}}{\partial t} = -(\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{B} + (\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{v} - \mathbf{B}\,\boldsymbol{\nabla} \cdot \mathbf{v}$$

- advection of magnetic field
- amplification by shear (stretching of field lines)
- amplification through compression

Dynamos: Motivation

- For $\mathbf{v} = 0$ magnetic field decays on timescale $\tau_d \sim L^2/\eta$
- Earth and other planets:
  - Evidence for magnetic field on earth for $3.5 \cdot 10^9$ years while $\tau_d \sim 10^4$ years
  - Permanent rock magnetism not possible since $T > T_{\mathrm{Curie}}$ and field highly variable $\longrightarrow$ field must be maintained by active process
- Sun and other stars:
  - Evidence for solar magnetic field for $\sim 300\,000$ years (^{10}Be)
  - Most solar-like stars show magnetic activity independent of age
  - Indirect evidence for stellar magnetic fields over life time of stars
  - But $\tau_d \sim 10^9$ years!
  - Primordial field could have survived in radiative interior of sun, but convection zone has much shorter diffusion time scale ~ 10 years (turbulent diffusivity)

Advection, diffusion, magnetic Reynolds number

Object	$\eta[\mathrm{m}^2/\mathrm{s}]$	$L[\mathrm{m}]$	$U[\mathrm{m/s}]$	R_m	τ_d
earth (outer core)	2	10^6	10^{-3}	300	10^4 years
sun (plasma conductivity)	1	10^8	100	10^{10}	10^9 years
sun (turbulent conductivity)	10^8	10^8	100	100	3 years
liquid sodium lab experiment	0.1	1	10	100	10 s

Mathematical definition of dynamo

S bounded volume with the surface ∂S, $\mathbf{B}$ maintained by currents contained within S, $B \sim r^{-3}$ asymptotically,

$$
\begin{aligned}
\frac{\partial \mathbf{B}}{\partial t} &= \nabla \times (\mathbf{v} \times \mathbf{B} - \eta \nabla \times \mathbf{B}) && \text{in } S \\
\nabla \times \mathbf{B} &= 0 && \text{outside } S \\
[\mathbf{B}] &= 0 && \text{across } \partial S \\
\nabla \cdot \mathbf{B} &= 0
\end{aligned}
$$

$\mathbf{v} = 0$ outside S, $\mathbf{n} \cdot \mathbf{v} = 0$ on ∂S and

$$E_{\mathrm{kin}} = \int_S \frac{1}{2} \varrho \mathbf{v}^2 \, dV \leq E_{\mathrm{max}} \quad \forall \, t$$

$\mathbf{v}$ is a dynamo if an initial condition $\mathbf{B} = \mathbf{B}_0$ exists so that

$$E_{\mathrm{mag}} = \int_{-\infty}^{\infty} \frac{1}{2\mu_0} \mathbf{B}^2 \, dV \geq E_{\mathrm{min}} \quad \forall \, t$$

Large scale/small scale dynamos

Decompose the magnetic field into large scale part and small scale part (energy carrying scale of turbulence) $\mathbf{B} = \overline{\mathbf{B}} + \mathbf{B}'$:

$$E_{\mathrm{mag}} = \int \frac{1}{2\mu_0} \overline{\mathbf{B}}^2 \, dV + \int \frac{1}{2\mu_0} \overline{\mathbf{B}'^2} \, dV \; .$$

- Small scale dynamo: $\overline{\mathbf{B}}^2 \ll \overline{\mathbf{B}'^2}$
- Large scale dynamo: $\overline{\mathbf{B}}^2 \geq \overline{\mathbf{B}'^2}$

Almost all turbulent (chaotic) velocity fields are small scale dynamos for sufficiently large R_m, large scale dynamos require additional large scale symmetries (see second half of this lecture)

What means large/small in practice (Sun)?

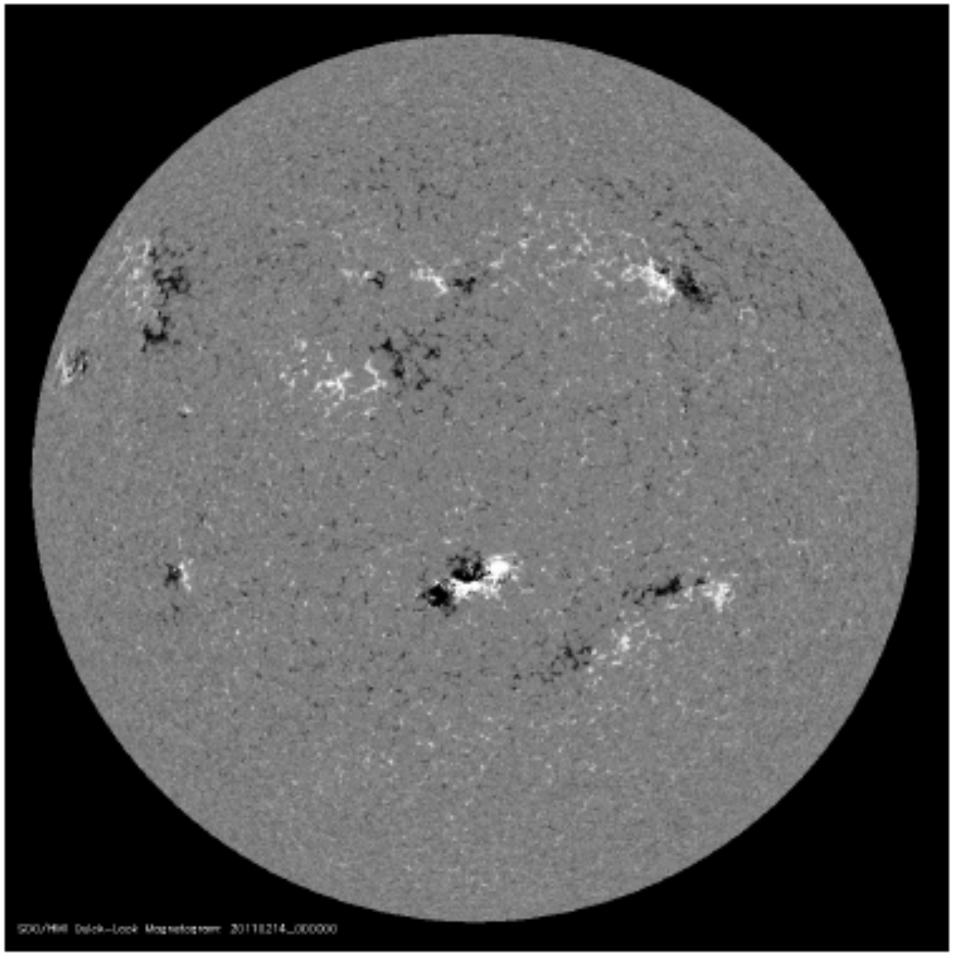
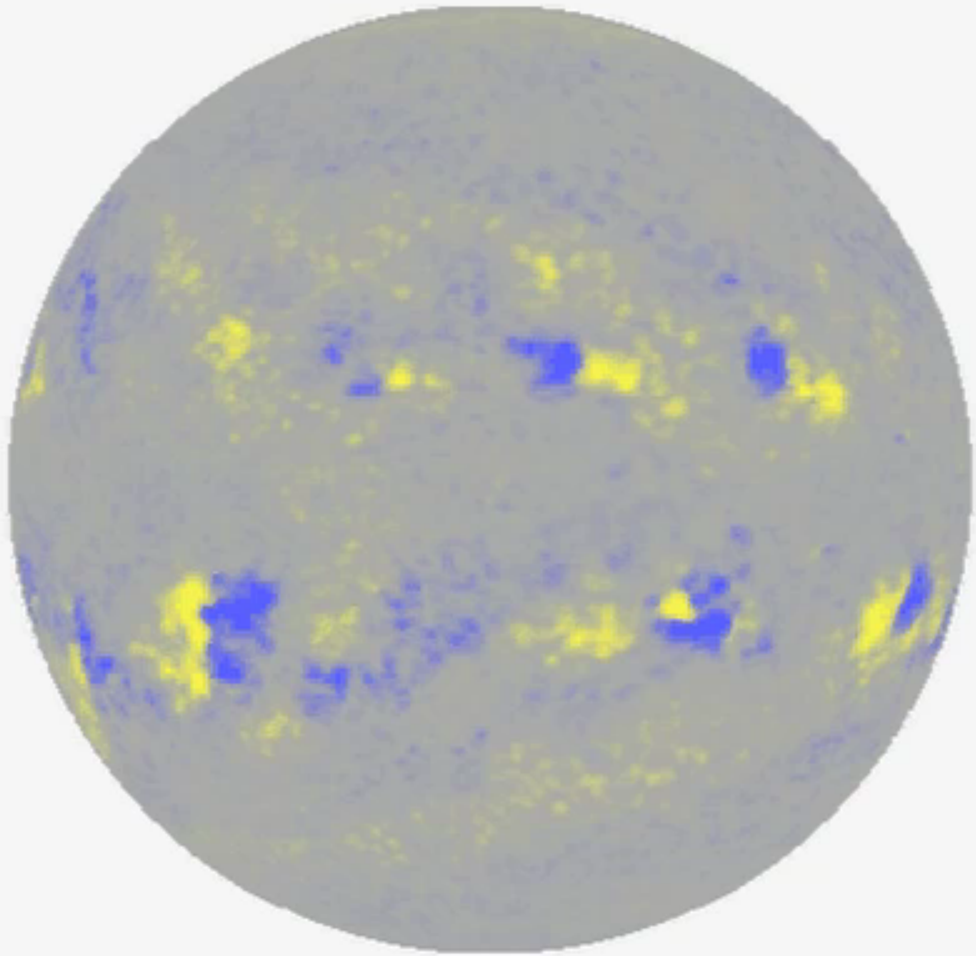
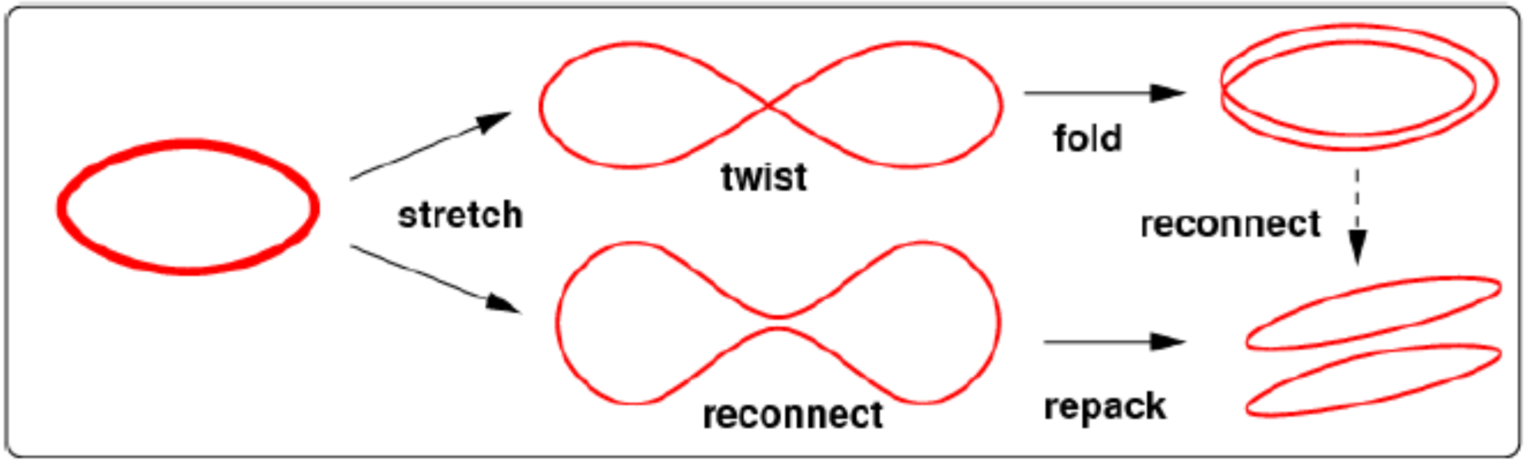


Figure: Full disk magnetogram SDO/HMI



Large scale/small scale dynamos



- Amplification through field line stretching
- Twist-fold required to repack field into original volume
- Magnetic diffusivity allows for change of topology

What is Magnetic Reconnection?

If a plasma is perfectly conducting, that is, it obeys the ideal Ohm's law,

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0$$

B-lines are frozen in the plasma, and no reconnection occurs.

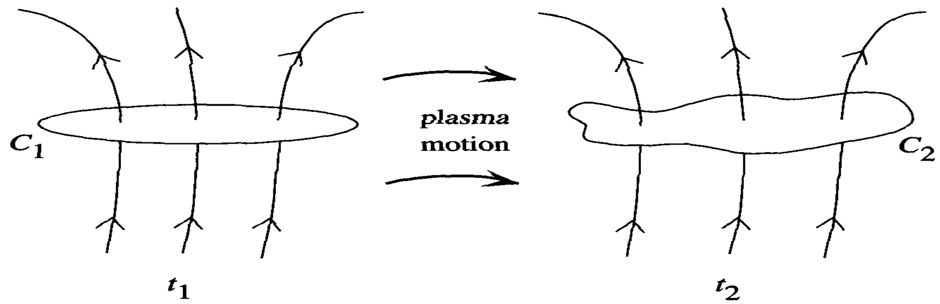


Fig. 1.6. Magnetic flux conservation: if a curve C_1 is distorted into C_2 by plasma motion, the flux through C_1 at t_1 equals the flux through C_2 at t_2.

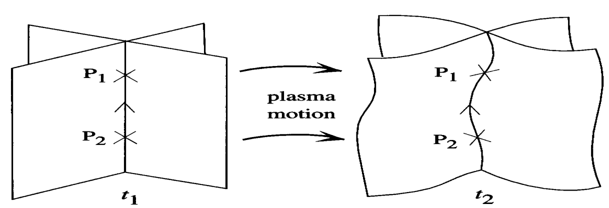


Fig. 1.7. Magnetic field-line conservation: if plasma elements P_1 and P_2 lie on a field line at time t_1, then they will lie on the same line at a later time t_2.

What is magnetic reconnection? (continued)

Departures from ideal behavior, represented by

$$\mathbf{E} + \mathbf{v} \times \mathbf{B}/c = \mathbf{R}, \quad \nabla \times \mathbf{R} \neq \mathbf{0}$$

break ideal topological invariants, allowing field lines to reconnect.

In the generalized Ohm's law for weakly collisional or collisionless plasmas, **R** contains resistivity, Hall current, electron inertia and pressure.

Slow/fast dynamos

Influence of magnetic diffusivity on growth rate

- Fast dynamo: growth rate independent of R_m (stretch-twist-fold mechanism)
- Slow dynamo: growth rate limited by resistivity (stretch-reconnect-repack)

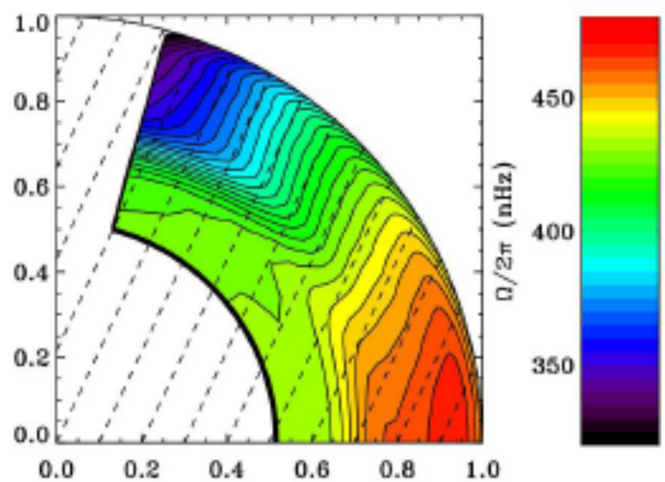
- Fast dynamos relevant for most astrophysical objects since $R_m \gg 1$
- Dynamos including (resistive) reconnection steps can be fast provided the reconnection is fast

Differential rotation and meridional flow

Induction effects of axisymmetric flows on axisymmetric field:

$$\mathbf{B} = B\mathbf{e_\Phi} + \nabla \times (A\mathbf{e_\Phi})$$
$$\mathbf{v} = v_r\mathbf{e_r} + v_\theta\mathbf{e_\theta} + \Omega\, r \sin\theta \mathbf{e_\Phi}$$

Differential rotation most dominant shear flow in stellar convection zones:



Meridional flow by-product of DR, observed as poleward surface flow

Differential rotation and meridional flow

Spherical geometry:

$$\frac{\partial B}{\partial t} + \frac{1}{r}\left(\frac{\partial}{\partial r}(r v_r B) + \frac{\partial}{\partial \theta}(v_\theta B)\right) =$$

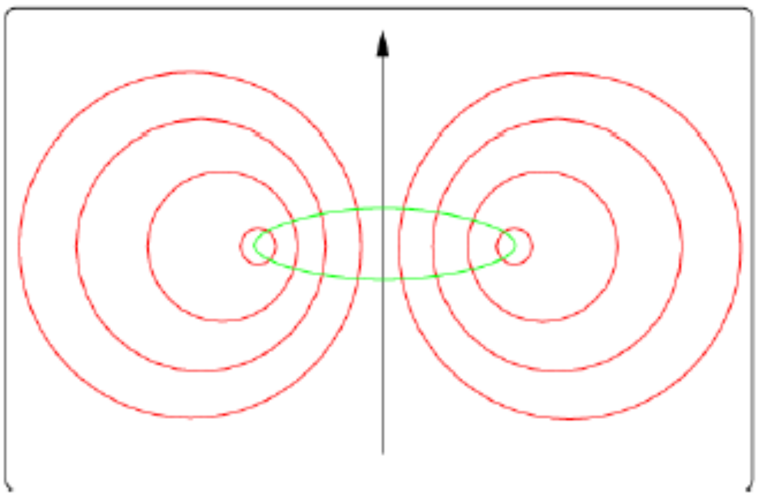
$$r \sin \mathbf{B}_p \cdot \nabla\Omega + \eta\left(\Delta - \frac{1}{(r\sin\theta)^2}\right) B$$

$$\frac{\partial A}{\partial t} + \frac{1}{r\sin\theta}\mathbf{v}_p \cdot \nabla(r\sin\theta A) = \eta\left(\Delta - \frac{1}{(r\sin\theta)^2}\right) A$$

- Meridional flow: Independent advection of poloidal and toroidal field
- Differential rotation: Source for toroidal field (if poloidal field not zero)
- Diffusion: Sink for poloidal and toroidal field
- No term capable of maintaining poloidal field against Ohmic decay!

Cowling's anti-dynamo theorem

A stationary axisymmetric magnetic field with currents limited to a finite volume in space cannot be maintained by a velocity field with finite amplitude.



Ohm's law of the form $\mathbf{j} = \sigma \mathbf{E}$ only decaying solutions, focus here on $\mathbf{j} = \sigma(\mathbf{v} \times \mathbf{B})$.
On O-type neutral line $\mathbf{B}_p$ is zero, but $\mu_0 \mathbf{j}_t = \nabla \times \mathbf{B}_p$ has finite value, but cannot be maintained by $(\mathbf{v} \times \mathbf{B})_t = (\mathbf{v}_p \times \mathbf{B}_p)$.

Meanfield induction equation

Average of induction equation:

$$\frac{\partial \overline{\mathbf{B}}}{\partial t} = \nabla \times \left(\overline{\mathbf{v}' \times \mathbf{B}'} + \overline{\mathbf{v}} \times \overline{\mathbf{B}} - \eta \nabla \times \overline{\mathbf{B}} \right)$$

New term resulting from small scale effects:

$$\overline{\mathcal{E}} = \overline{\mathbf{v}' \times \mathbf{B}'}$$

Fluctuating part of induction equation:

$$\left(\frac{\partial}{\partial t} - \eta \Delta \right) \mathbf{B}' - \nabla \times (\overline{\mathbf{v}} \times \mathbf{B}') = \nabla \times \left(\mathbf{v}' \times \overline{\mathbf{B}} + \mathbf{v}' \times \mathbf{B}' - \overline{\mathbf{v}' \times \mathbf{B}'} \right)$$

Kinematic approach: $\mathbf{v}'$ assumed to be given

- Solve for $\mathbf{B}'$, compute $\overline{\mathbf{v}' \times \mathbf{B}'}$ and solve for $\overline{\mathbf{B}}$
- Term $\mathbf{v}' \times \mathbf{B}' - \overline{\mathbf{v}' \times \mathbf{B}'}$ leading to higher order correlations (closure problem)

Mean field expansion of turbulent induction effects

Exact expressions for $\overline{\boldsymbol{\mathcal{E}}}$ exist only under strong simplifying assumptions (see homework assignment).

In general $\overline{\boldsymbol{\mathcal{E}}}$ is a linear functional of $\overline{\mathbf{B}}$:

$$\overline{\mathcal{E}}_i(\mathbf{x},t) = \int_{-\infty}^{\infty} d^3x' \int_{-\infty}^{t} dt' \, \mathcal{K}_{ij}(\mathbf{x},t,\mathbf{x}',t') \, \overline{B}_j(\mathbf{x}',t') \; .$$

Can be simplified if a sufficient scale separation is present:

- $l_c \ll L$
- $\tau_c \ll \tau_L$

Leading terms of expansion:

$$\overline{\mathcal{E}}_i = a_{ij} \overline{B}_j + b_{ijk} \frac{\partial \overline{B}_j}{\partial x_k}$$

Symmetry constraints

Decomposing a_{ij} and $\partial\overline{B}_j/\partial x_k$ into symmetric and antisymmetric components:

$$a_{ij} = \underbrace{\frac{1}{2}(a_{ij}+a_{ji})}_{\alpha_{ij}} + \underbrace{\frac{1}{2}(a_{ij}-a_{ji})}_{-\varepsilon_{ijk}\gamma_k}$$

$$\frac{\partial\overline{B}_j}{\partial x_k} = \frac{1}{2}\left(\frac{\partial\overline{B}_j}{\partial x_k}+\frac{\partial\overline{B}_k}{\partial x_j}\right) + \underbrace{\frac{1}{2}\left(\frac{\partial\overline{B}_j}{\partial x_k}-\frac{\partial\overline{B}_k}{\partial x_j}\right)}_{-\frac{1}{2}\varepsilon_{jkl}(\nabla\times\overline{\mathbf{B}})_l}$$

Leads to:

$$\overline{\mathcal{E}}_i = \alpha_{ik}\overline{B}_k + \varepsilon_{ijk}\gamma_j\overline{B}_k - \underbrace{\frac{1}{2}b_{ijk}\varepsilon_{jkl}}_{\beta_{il}-\varepsilon_{ilm}\delta_m}(\nabla\times\overline{\mathbf{B}})_l + \ldots$$

Symmetry constraints

Overall result:

$$\overline{\mathcal{E}} = \alpha \overline{\mathbf{B}} + \gamma \times \overline{\mathbf{B}} - \beta \nabla \times \overline{\mathbf{B}} - \delta \times (\nabla \times \overline{\mathbf{B}}) + \ldots$$

With:

$$\alpha_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) \;, \qquad \gamma_i = -\frac{1}{2}\varepsilon_{ijk} a_{jk}$$

$$\beta_{ij} = \frac{1}{4}(\varepsilon_{ikl} b_{jkl} + \varepsilon_{jkl} b_{ikl}) \;, \qquad \delta_i = \frac{1}{4}(b_{jji} - b_{jij})$$

Simplified expressions

Assuming $|\mathbf{B}'| \ll |\overline{\mathbf{B}}|$ in derivation + additional simplification for (quasi) isotropic, non-mirror symmetric, (weakly) inhomogeneous turbulence:

$$\overline{v_i' v_j'} \sim \delta_{ij},\ \alpha_{ij} = \alpha\delta_{ij},\ \beta_{ij} = \eta_t\delta_{ij}$$

Leads to:

$$\frac{\partial \overline{\mathbf{B}}}{\partial t} = \boldsymbol{\nabla} \times \left[\alpha\overline{\mathbf{B}} + (\overline{\mathbf{v}} + \gamma) \times \overline{\mathbf{B}} - (\eta + \eta_t)\, \boldsymbol{\nabla} \times \overline{\mathbf{B}}\right]$$

with the scalar quantities

$$\alpha = -\frac{1}{3}\tau_c\, \overline{\mathbf{v}' \cdot (\boldsymbol{\nabla} \times \mathbf{v}')}, \quad \eta_t = \frac{1}{3}\tau_c\, \overline{\mathbf{v}'^2}$$

and vector

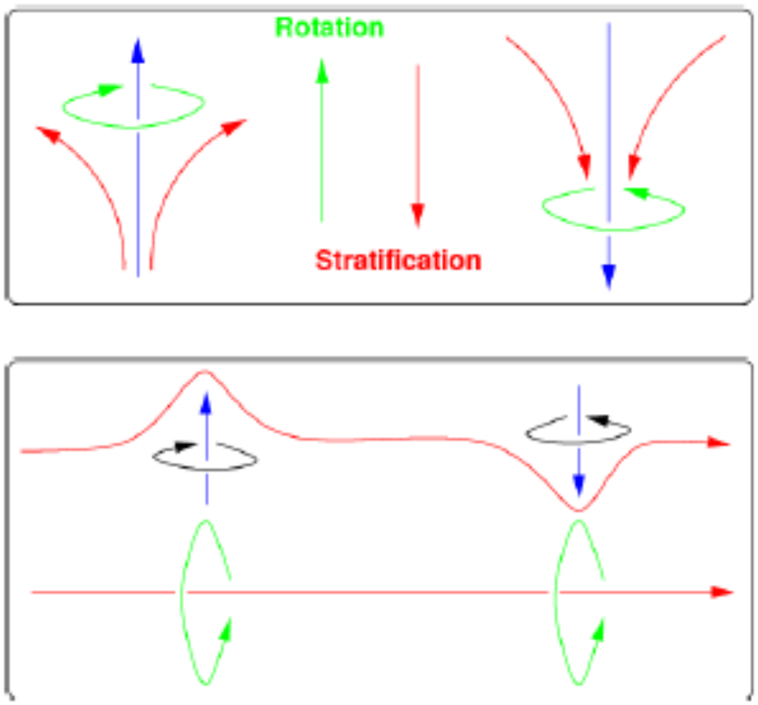
$$\gamma = -\frac{1}{6}\tau_c \boldsymbol{\nabla}\overline{\mathbf{v}'^2} = -\frac{1}{2}\boldsymbol{\nabla}\eta_t$$

Expressions are independent of η (in this approximation): fast dynamo

Kinematic α-effect

$$\alpha = -\frac{1}{3}\tau_c \overline{\mathbf{v}' \cdot (\nabla \times \mathbf{v}')} \quad H_k = \overline{\mathbf{v}' \cdot (\nabla \times \mathbf{v}')} \text{ kinetic helicity}$$

Requires rotation + additional preferred direction (stratification)



$\alpha\Omega$-dynamo

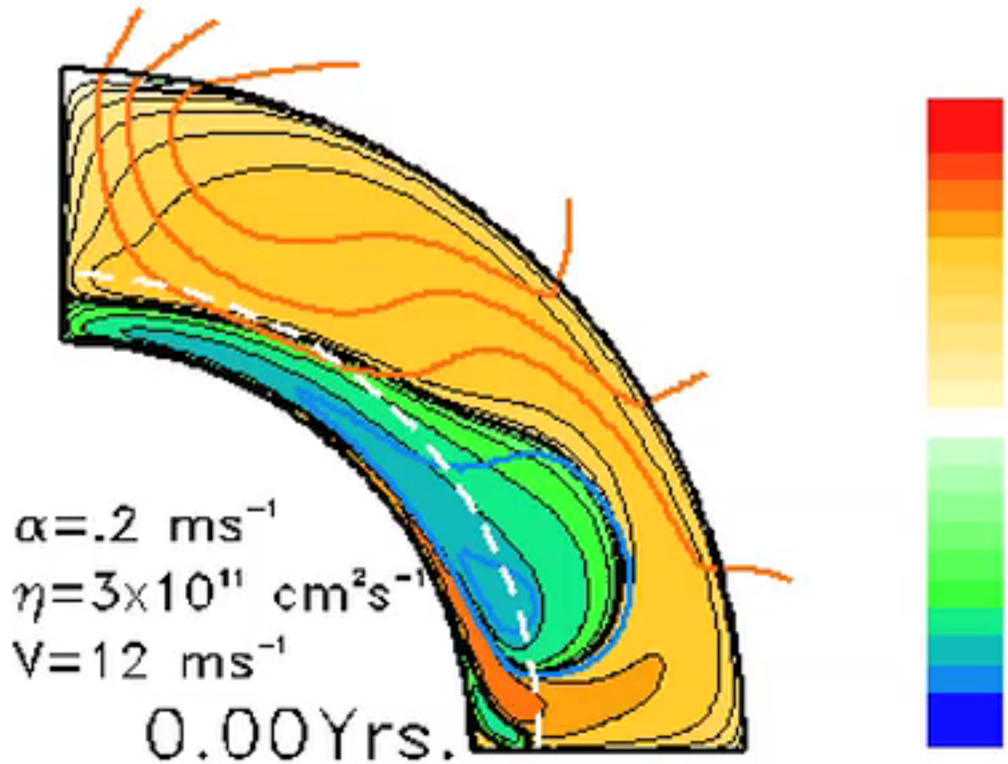
$$\frac{\partial B}{\partial t} + \frac{1}{r}\left(\frac{\partial}{\partial r}(r v_r B) + \frac{\partial}{\partial \theta}(v_\theta B)\right) = r \sin \mathbf{B_p} \cdot \boldsymbol{\nabla}\Omega$$
$$+ \eta\left(\Delta - \frac{1}{(r\sin\theta)^2}\right) B$$
$$\frac{\partial A}{\partial t} + \frac{1}{r\sin\theta}\mathbf{v}_p \cdot \boldsymbol{\nabla}(r\sin\theta A) = \alpha B + \eta\left(\Delta - \frac{1}{(r\sin\theta)^2}\right) A$$

- Dimensionless measure for strength of Ω- and α-effect

$$D_\Omega = \frac{R^2\Delta\Omega}{\eta_t} \qquad D_\alpha = \frac{R\alpha}{\eta_t}$$

- Dynamo excited if dynamo number

$$D = D_\Omega D_\alpha > D_{crit}$$



α=.2 ms⁻¹
η=3x10¹¹ cm²s⁻¹
V=12 ms⁻¹
0.00Yrs.

Non-kinematic effects

Proper way to treat them: 3D simulations

- Still very challenging
- Has been successful for geodynamo, but not for solar dynamo

Semi-analytical treatment of Lorentz-force feedback in mean field models:

- Macroscopic feedback: Change of the mean flow (differential rotation, meridional flow) through the mean Lorentz-force

$$\overline{\mathbf{f}} = \overline{\mathbf{j}} \times \overline{\mathbf{B}} + \overline{\mathbf{j}' \times \mathbf{B}'}$$

- Mean field model including mean field representation of full MHD equations

- Microscopic feedback: Change of turbulent induction effects (e.g. α-quenching)

Microscopic feedback

Symmetry of momentum and induction equation $\mathbf{v}' \leftrightarrow \mathbf{B}'/\sqrt{\mu_0 \varrho}$:

$$\begin{aligned}
\frac{d\mathbf{v}'}{dt} &= \frac{1}{\mu_0 \varrho}(\overline{\mathbf{B}} \cdot \nabla)\mathbf{B}' + \dots \\
\frac{d\mathbf{B}'}{dt} &= (\overline{\mathbf{B}} \cdot \nabla)\mathbf{v}' + \dots \\
\overline{\mathcal{E}} &= \overline{\mathbf{v}' \times \mathbf{B}'}
\end{aligned}$$

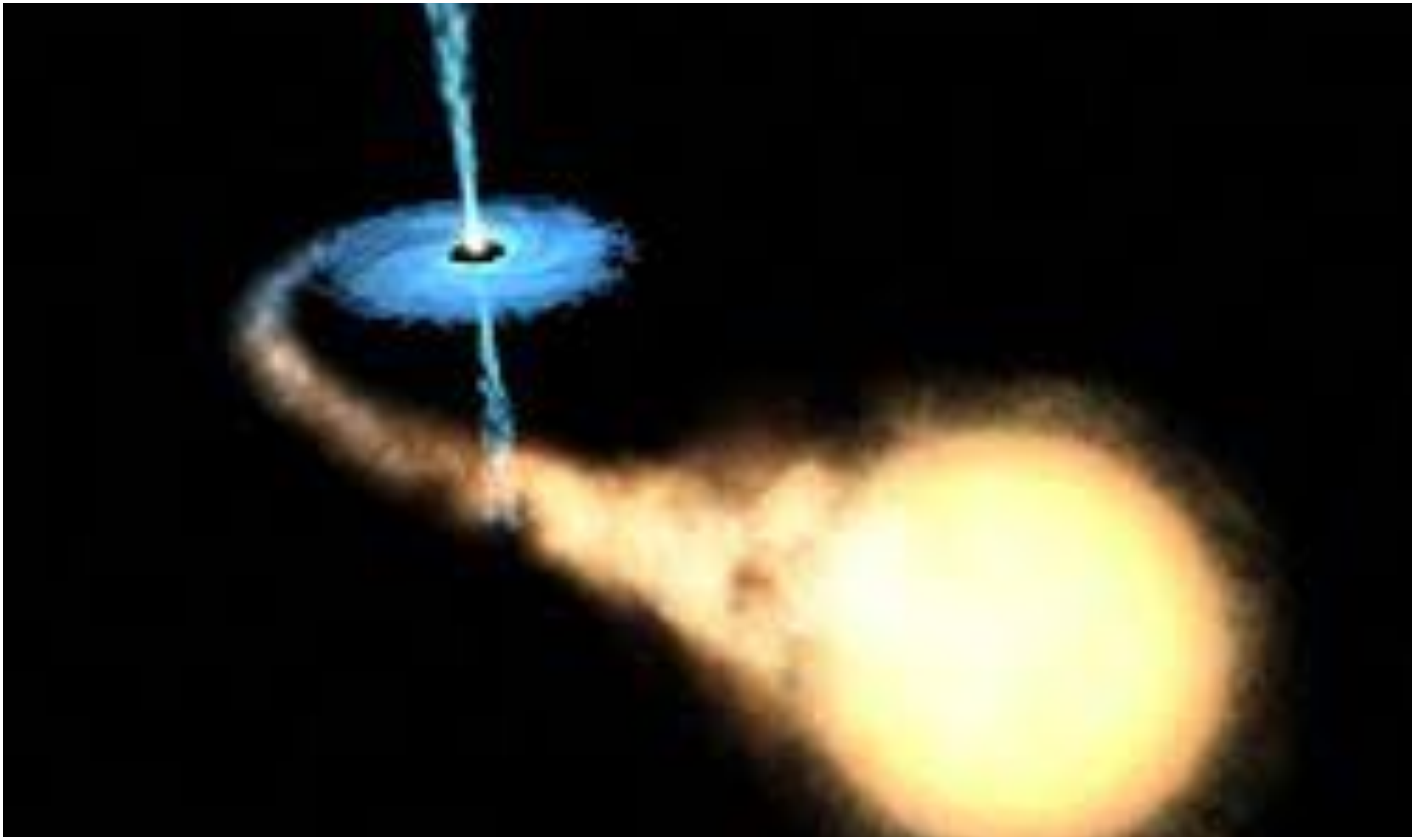
Strongly motivates magnetic term for α-effect (Pouquet et al. 1976):

$$\alpha = \frac{1}{3}\tau_c \left(\frac{1}{\varrho} \overline{\mathbf{j}' \cdot \mathbf{B}'} - \overline{\omega' \cdot \mathbf{v}'} \right)$$

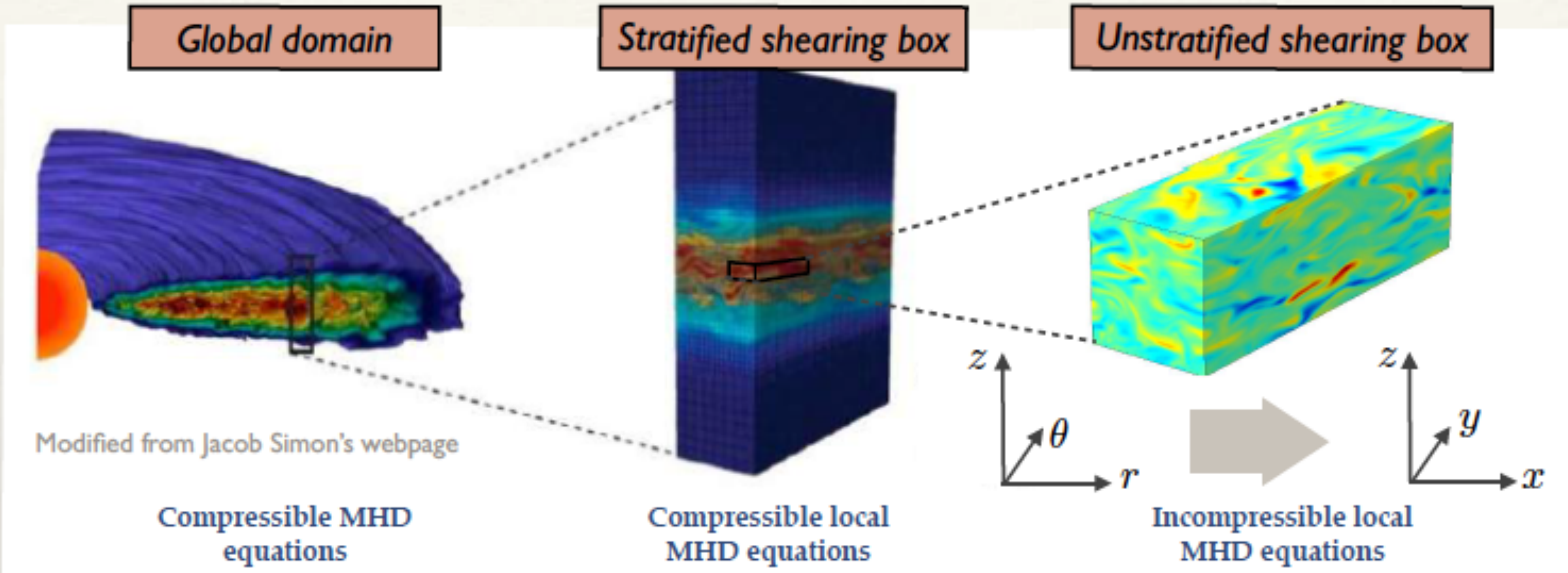
Challenges to Kinematic Mean-Field Dynamo Theory

- Smallest scales grows most rapidly (*Kulsrud and Anderson 1972, Boldyrev et al. 2005*)
- Due to constraints of magnetic helicity conservation, small-scale fields act back to decrease the large-scale field growth drastically---the problem of "catastrophic quenching" (*Gruzinov and Diamond 1994, Cattaneo and Hughes 2009*). But this challenge could be addressed by transporting helicity (*Blackman and Field 2002, Subramanian and Brandenburg 2004, Ebrahimi and B. 2014, Tobias and Cattaneo 2014*)
- At even moderate Rm, the fast-growing small-scale dynamo implies that velocity fluctuations should always be accompanied by magnetic field fluctuations of a similar magnitude (*Schekochihin et al. 2004*), questioning the relevance of the classical kinematic theory.

Accretion Disks near a Black Hole

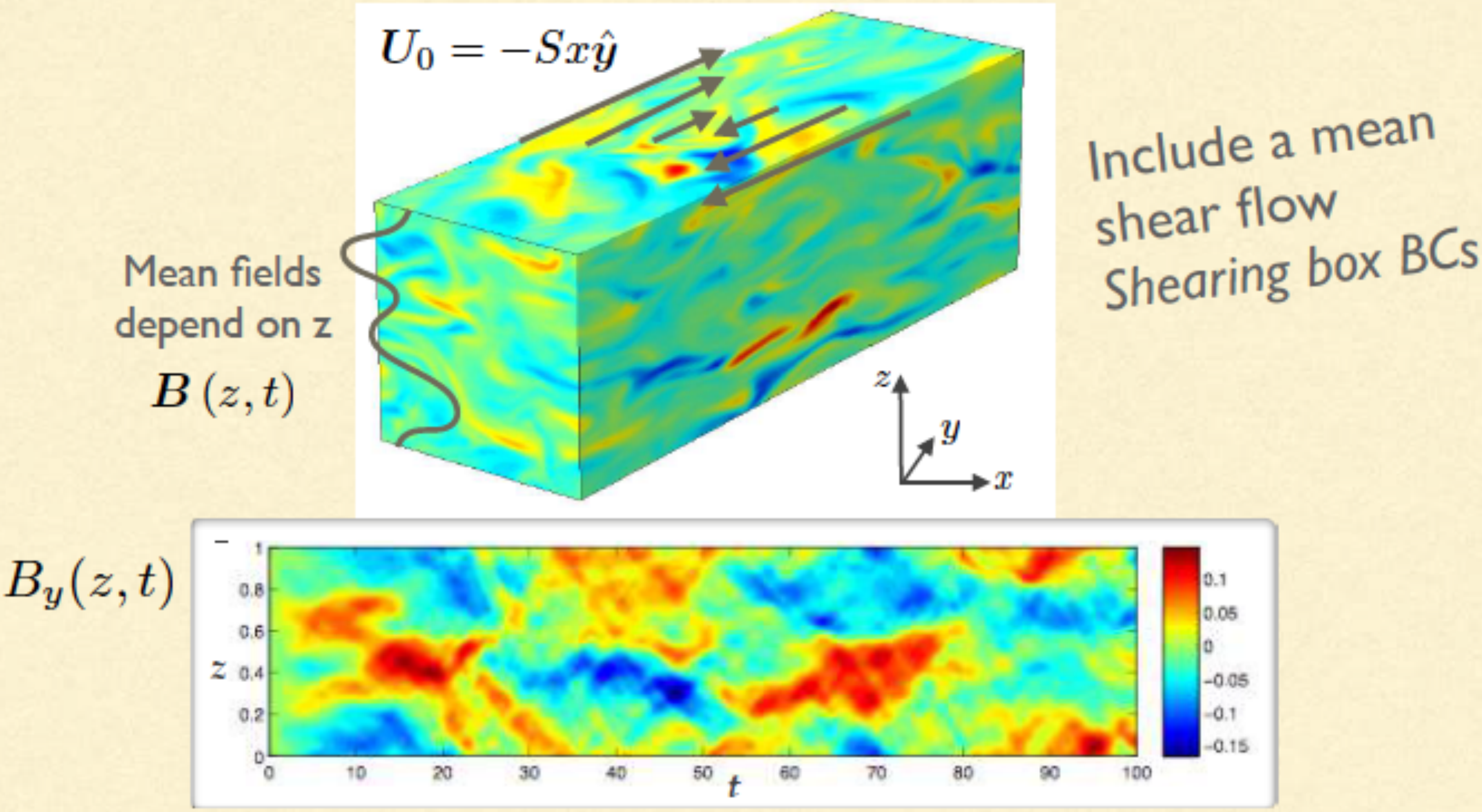


Preliminary – local MRI



- The simplest relevant system exhibiting MRI turbulence is the local incompressible MHD equations — remove global curvature
- In the *shearing box*, boundary conditions are periodic in y (azimuthal) and z (vertical), and shearing periodic in x (radial).

- We use the horizontal average for the mean-field average.



- Study the dynamo by studying $\mathcal{E}(B, U)$.

PRIMARY RESULT

New dynamo mechanism
— *the magnetic shear-current effect* —
small-scale magnetic fields have a *positive* effect on the large-scale dynamo.

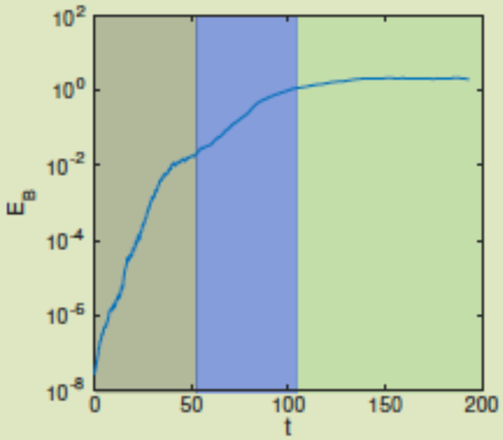
Effect requires velocity shear (e.g., Keplerian).

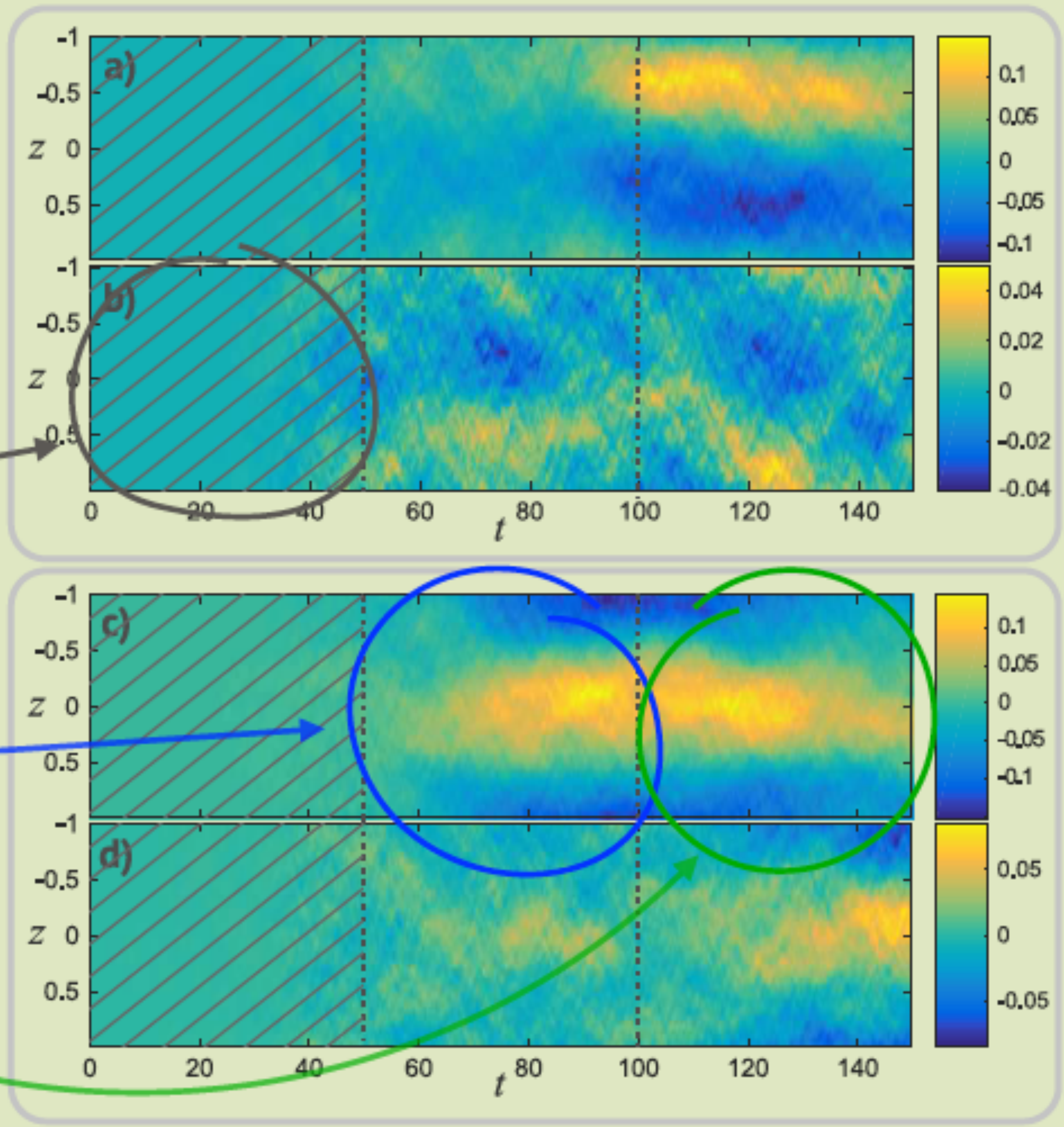
No α effect required.

Off-diagonal component of β couples with the shear.

Squire & Bhattacharjee PRL (2015) *Squire & Bhattacharjee ApJ (2015)* *Squire & Bhattacharjee PRE (2015)*

Rm=2000 — 100 realizations





$\Omega = 0$

Keplerian

Fast growth of small-scale dynamo, saturates t≈40.

Large-scale dynamo driven by small-scale *b* fluctuations?

Large-scale dynamo saturates — change in η?

- Magnetic shear-current effect is like *inverse* quenching — small-scale dynamo can *drive* a large-scale dynamo.

Agreement between simulation and analytic results.

- Good evidence that magnetic shear-current effect is responsible for unstratified MRI dynamo (Shi, Stone, and Huang 2016)

Shear flows being ubiquitous, is the magnetic shear-current effect important for the Sun? (Hotta, Rempel, and Yokoyama, 2016)