Basic Plasma Concepts and Models

Amitava Bhattacharjee
University of New Hampshire

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Goal of this lecture

- Review a few basic plasma concepts and models that underlie the lectures later in the week.
- There are several excellent text books in plasma physics: Chen, Nicholson (out of print), Goldston and Rutherford, Boyd and Sanderson, Bellan.
- The book I am most familiar with is by Gurnett and Bhattacharjee, from which most of the material is taken.

What is a Plasma?

Plasma is an ensemble of charged particles, capable of exhibiting collective interactions.

Levels of Description:

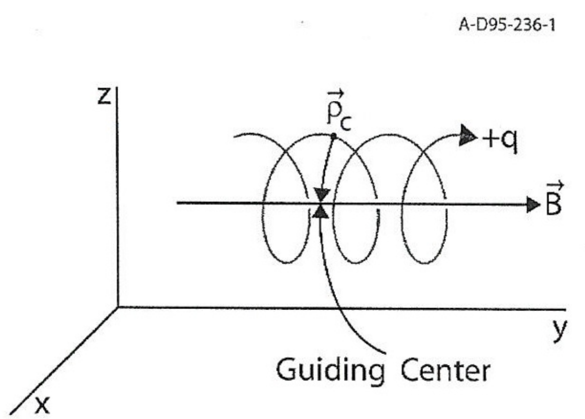
- Single-particle dynamics in prescribed electric and magnetic fields
- Plasmas as fluids in 3D configuration space moving under the influence of self-consistent electric and magnetic fields
- Plasmas as kinetic fluids in 6D μ-space (that is, configuration and velocity space), coupled to self-consistent Maxwell's equations.

Single-Particle Orbit Theory

Newton's law of motion for charged particles

$$m\frac{d\mathbf{v}}{dt} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$$

Guiding-Center: A very useful concept



Single-Particle Orbit Theory

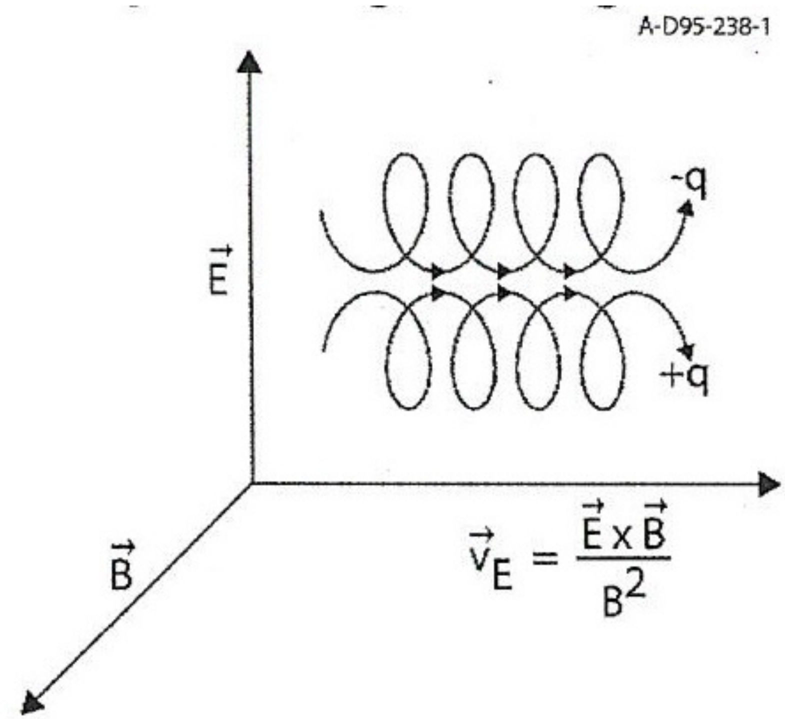
ExB Drift

$$m\frac{d\mathbf{v}}{dt} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$$

Consider $\mathbf{E} = const.,\ \mathbf{B} = const.$

The charged particles experience a drift velocity, perpendicular to both E and B, and independent of their charge and mass.

$$\mathbf{v}_{\mathbf{E}} = \frac{\mathbf{E} \times \mathbf{B}}{B^2}$$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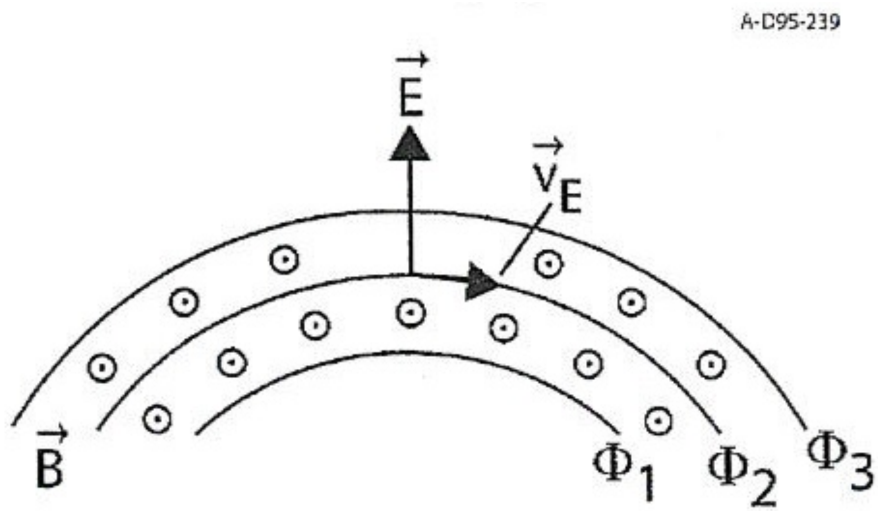
E
-q
+q
B
$\vec{v}_E = \frac{\vec{E} \times \vec{B}}{B^2}$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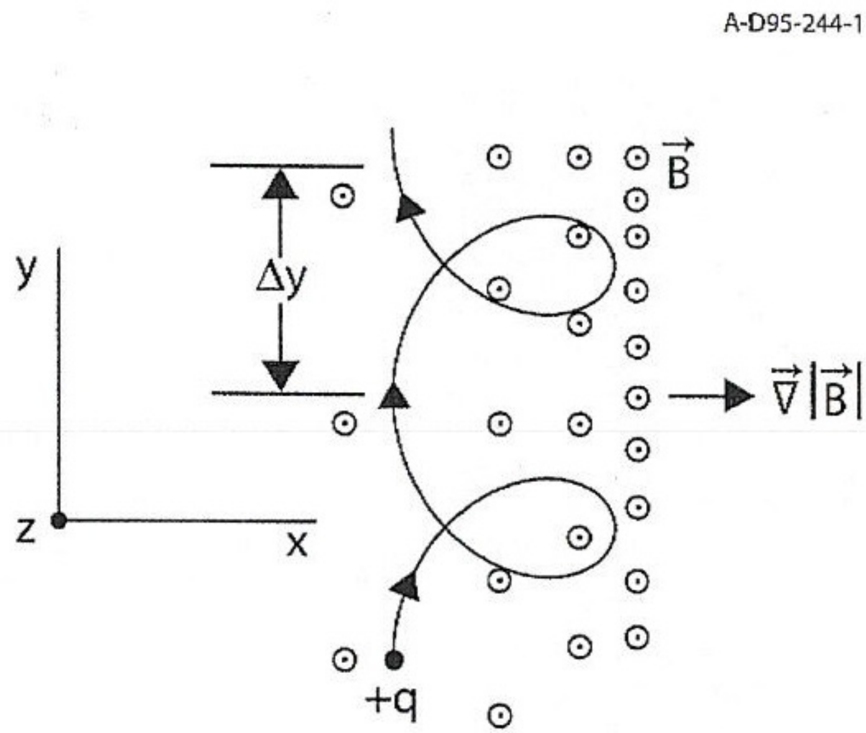
E
$\vec{v}_E$
B
Φ_1
Φ_2
Φ_3

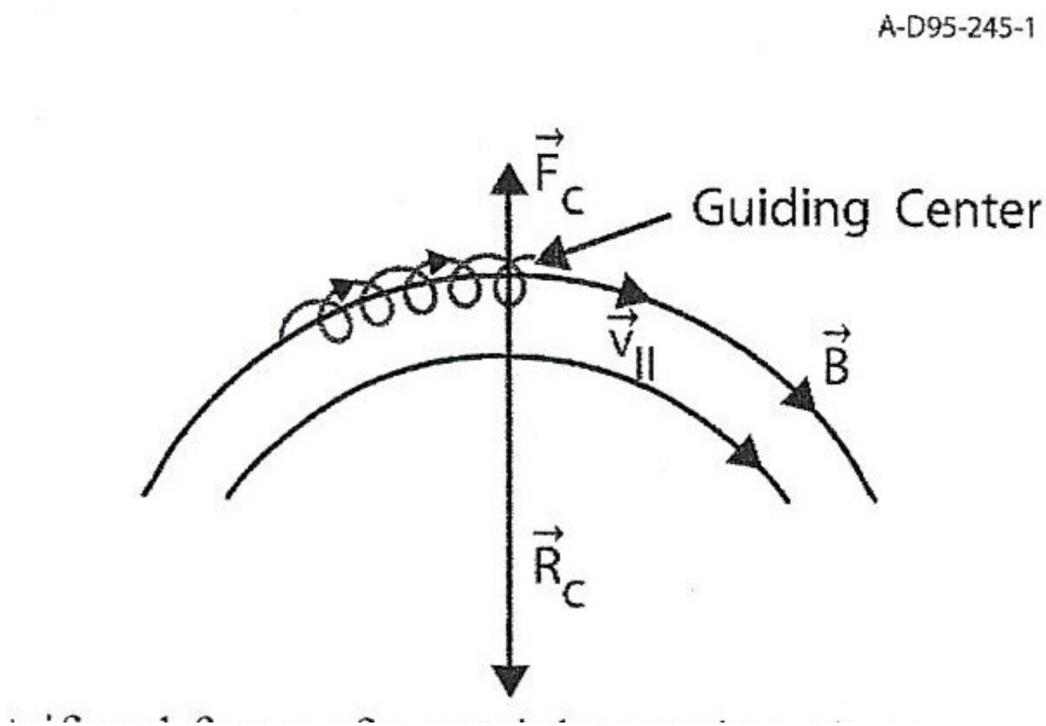
Gradient B drift



$$\mathbf{V_G} = \frac{w_\perp}{qB}\left(\frac{\mathbf{B}\times\nabla\mathbf{B}}{B^2}\right),$$

$$w_\perp = \frac{1}{2}{\omega_c}^2{\rho_c}^2$$

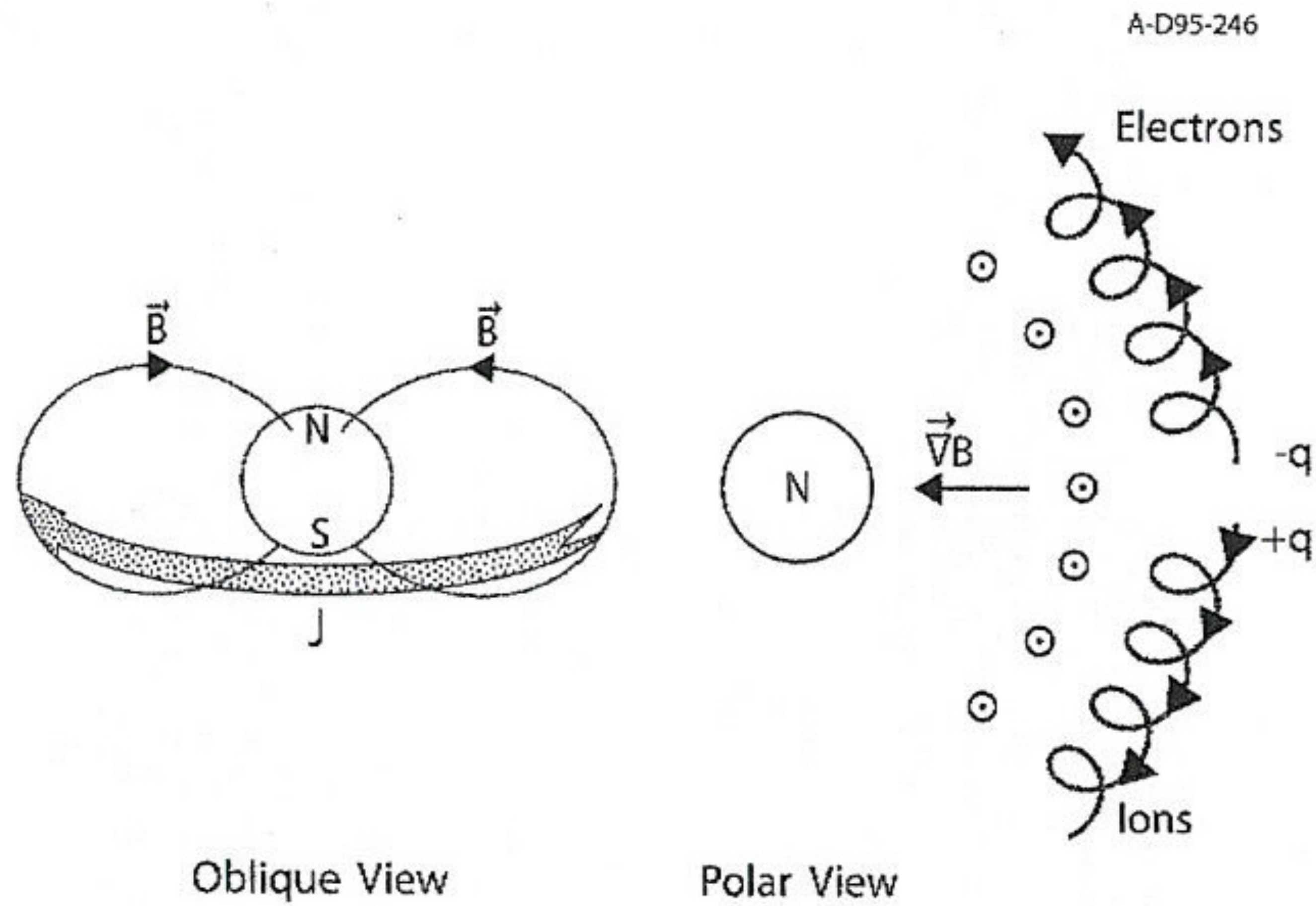
Curvature drift



$$\mathbf{V}_C = \frac{2w_{||}}{qB^2}\left(\frac{\mathbf{R_C} \times \mathbf{B}}{R_C{}^2}\right),$$

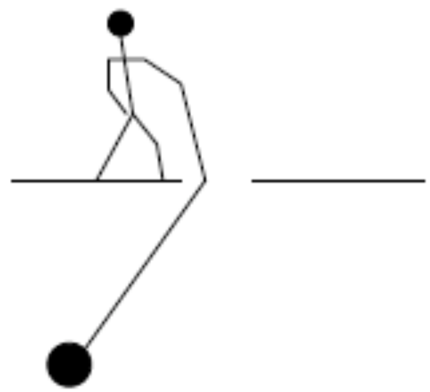
$$w_{||} = \frac{1}{2}mv_{||}{}^2$$

The Ring Current in Earth's Magnetosphere: An Example



Adiabatic Invariants

Albert Einstein and the Adiabatic Pendulum (1911)



Einstein suggested that while both the energy E and the frequency ν change, the ratio E/ν remains approximately invariant.

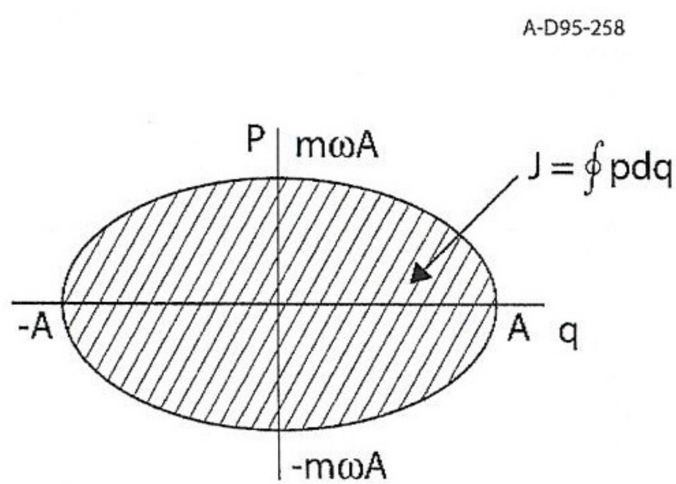
Adiabatic Invariants

Harmonic oscillator

$$\frac{d^2x}{dt^2} + \omega^2(\varepsilon t)x = 0,\ \varepsilon << 1$$

The adiabatic invariant is

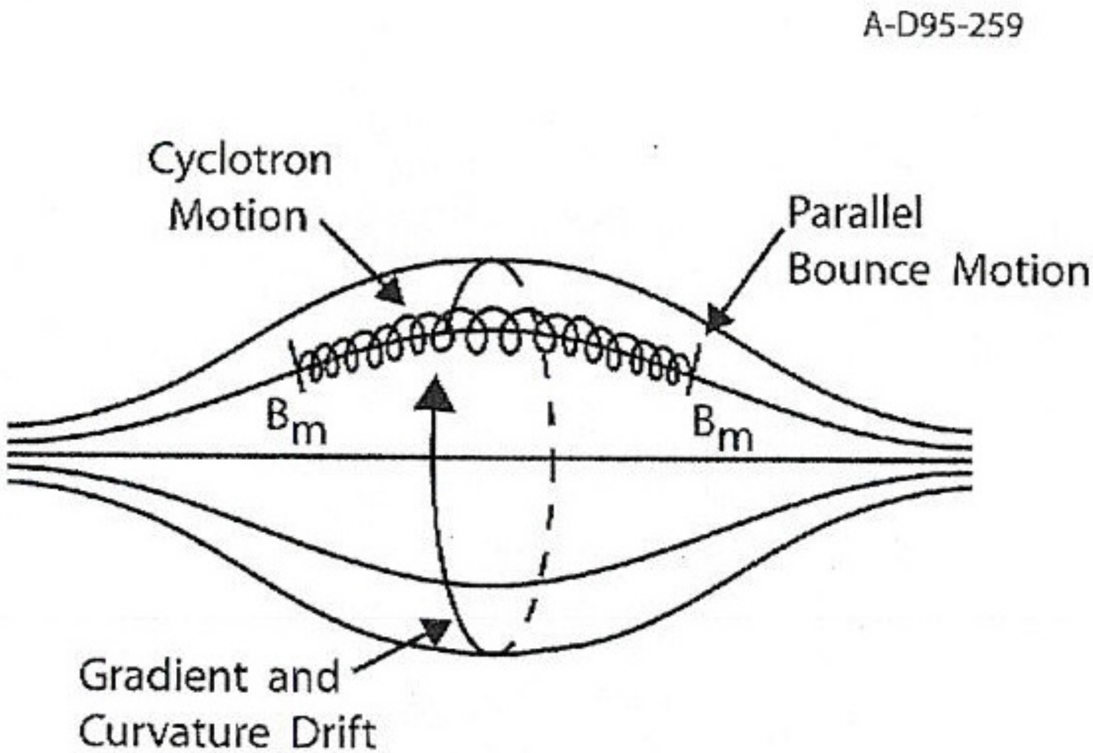
$$J = \oint p\,dq$$



$$\Delta J / J \sim \exp(-c/\varepsilon)$$

Adiabatic Invariants

Three types of bounce motion



Adiabatic Invariants

Three types of bounce motion

First adiabatic invariant $\mu = w_\perp / B$

Second adiabatic invariant $J = m \oint v_{||} ds$

Third adiabatic invariant $\Phi = \pi R^2 B$

Kinetic Description of Plasmas

Distribution function $f(\mathbf{r},\mathbf{v},t)$

Normalization $N = \iint\limits_{phase\ space} d\mathbf{x}d\mathbf{v}f(\mathbf{r},\mathbf{v},t)$

Example: Maxwell distribution function

$$f = n_0 \exp\left(-\frac{mv^2}{2kT}\right), \; n_0 = N/V$$

Boltzmnn-Vlasov Equation

Motion of an incompressible phase fluid
in μ−space

$$\frac{\partial f_s}{\partial t} + \mathbf{v}.\frac{\partial f_s}{\partial t} + \frac{q}{m}\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right).\frac{\partial f_s}{\partial \mathbf{v}} = 0,\ s = e, i$$

In the presence of collisions

$$\frac{\partial f_s}{\partial t} + \mathbf{v}.\frac{\partial f_s}{\partial t} + \frac{q}{m}\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right).\frac{\partial f_s}{\partial \mathbf{v}} = \left(\frac{\delta f_s}{\delta t}\right)_c$$

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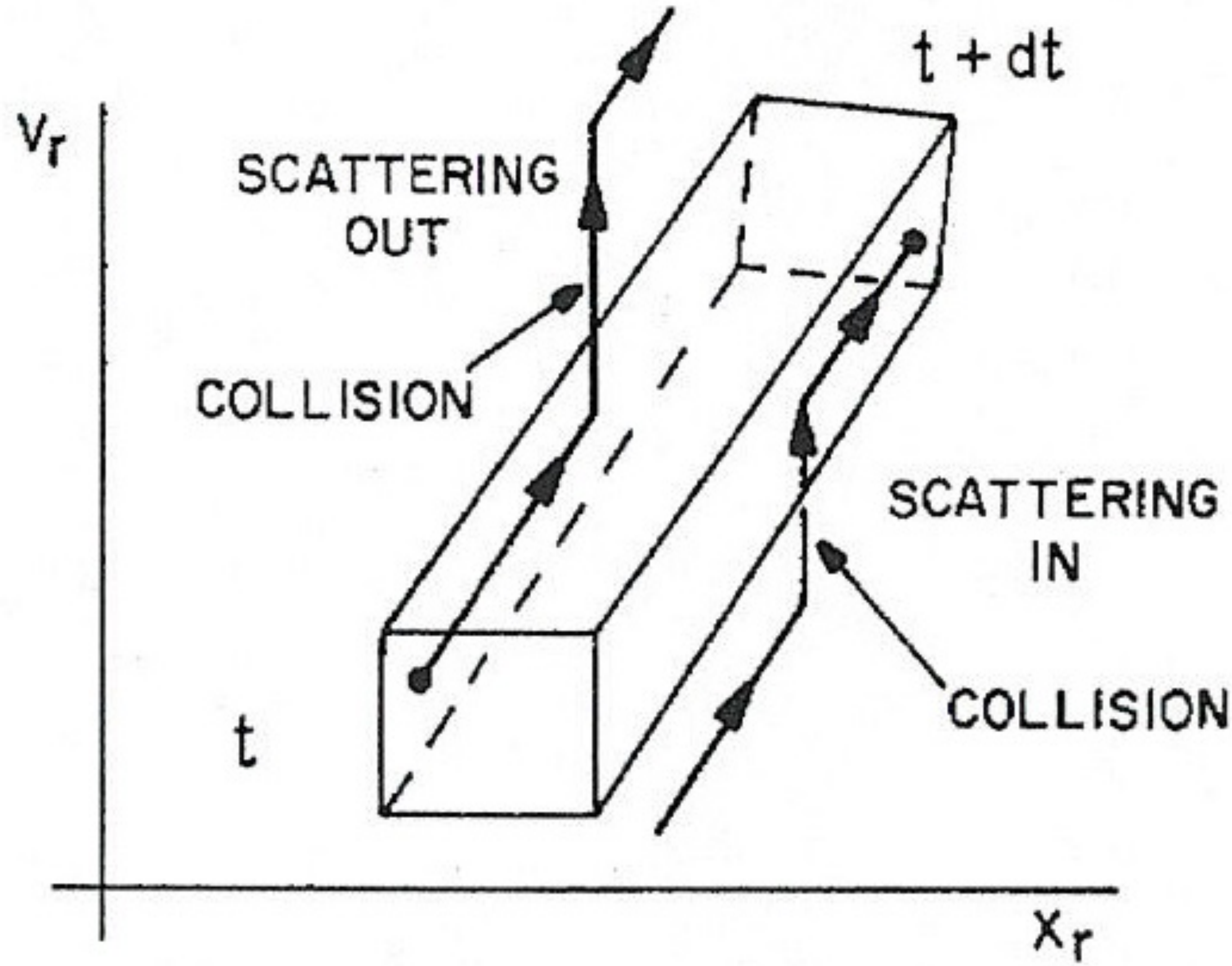


t + dt
V_r
SCATTERING
OUT
COLLISION
SCATTERING
IN
COLLISION
t
X_r

Vlasov-Poisson equations: requirements of self-consistency in an electrostatic plasma

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \frac{\partial f_s}{\partial \mathbf{r}} - \frac{q_s}{m_s} \nabla \Phi \cdot \frac{\partial f_s}{\partial \mathbf{v}} = 0$$

$$\mathbf{E} = -\nabla \Phi$$

$$\nabla \cdot \mathbf{E} = -\nabla^2 \Phi = 4\pi\rho = 4\pi \sum_s q_s \int d\mathbf{v} f_s$$

Quasilinear theory: application to scattering due to wave-particle interactions

- Consider electrostatic Vlasov equation

$$\frac{\partial f_s}{\partial t} + \mathbf{v}.\frac{\partial f_s}{\partial t} - \frac{q}{m}\nabla \boldsymbol{\Phi}.\frac{\partial f_s}{\partial \mathbf{v}} = 0.$$

Split every dependent variable into a mean and a fluctuation

$$f_s = \langle f_s \rangle + f_{s1}\,,\ \langle f_{s1} \rangle = 0$$

Quasilinear Diffusion

It follows after some algebra that the mean or average distribution function obeys a diffusion equation:

$$\frac{\partial}{\partial t}\langle f_s \rangle = \frac{\partial}{\partial \mathbf{v}} \cdot \left(\mathbf{D} \cdot \frac{\partial}{\partial \mathbf{v}} \langle f_s \rangle \right)$$

Here D is a diffusion tensor, dependent on wave fluctuations (pertinent to Shprits lecture).

Lecture for which this material is directly pertinent

- *Shprits* : Radiation Belts
- *Sojka* : The Ionosphere

Fluid Models

The primary fluid model of focus in this summer school is Magnetohydrodynamics (MHD)

It treats the plasma as a single fluid, without distinguishing between electrons or protons, moving under the influence of self-consistent electric and magnetic fields.

It can be derived from kinetic theory by taking moments (integrating over velocity space), and making some drastic approximations.

Equations of MHD

Newton

$$\frac{m\mathbf{a}}{vol} = \frac{\mathbf{F}}{vol} \longrightarrow \rho \frac{d\mathbf{v}}{dt} = -\nabla P + \mathbf{J} \times \mathbf{B}$$

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}\right) = -\nabla P + \mathbf{J} \times \mathbf{B}$$

Ohm's Law:

$$V = IR \longrightarrow \mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{J}$$

Maxwell's equations:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \quad \text{and} \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

MHD Equations (continued)

- Ohm's law: $\vec{j} = \sigma\left(\vec{E} + \vec{u} \times \vec{B}\right)$

  Faraday's and Ohm's law

$$\begin{aligned}\frac{\partial \vec{B}}{\partial t} - \nabla \times (\vec{u} \times \vec{B}) &= -\frac{1}{\sigma}\nabla \times \vec{j} \\ &= -\frac{1}{\sigma}\nabla \times \frac{\nabla \times \vec{B}}{\mu_o} = -\frac{1}{\mu_o \sigma}\nabla \times \left(\nabla \times \vec{B}\right) \\ &= \frac{1}{\mu_o \sigma}\nabla^2 \vec{B}\end{aligned}$$

  Frozen-in magnetic field: $\sigma \to \infty$

- Equation of state:

  Often the weakest link in MHD theory, such as the isothermal or adiabatic assumption, $p / \rho^{\gamma} = const.$

Static, force-balanced equilibria

- In ideal MHD equations (that is, MHD equations without dissipation), consider steady (or quasi-steady) states

$$\frac{\partial}{\partial t} = 0, \ \mathbf{v} = \mathbf{0}$$

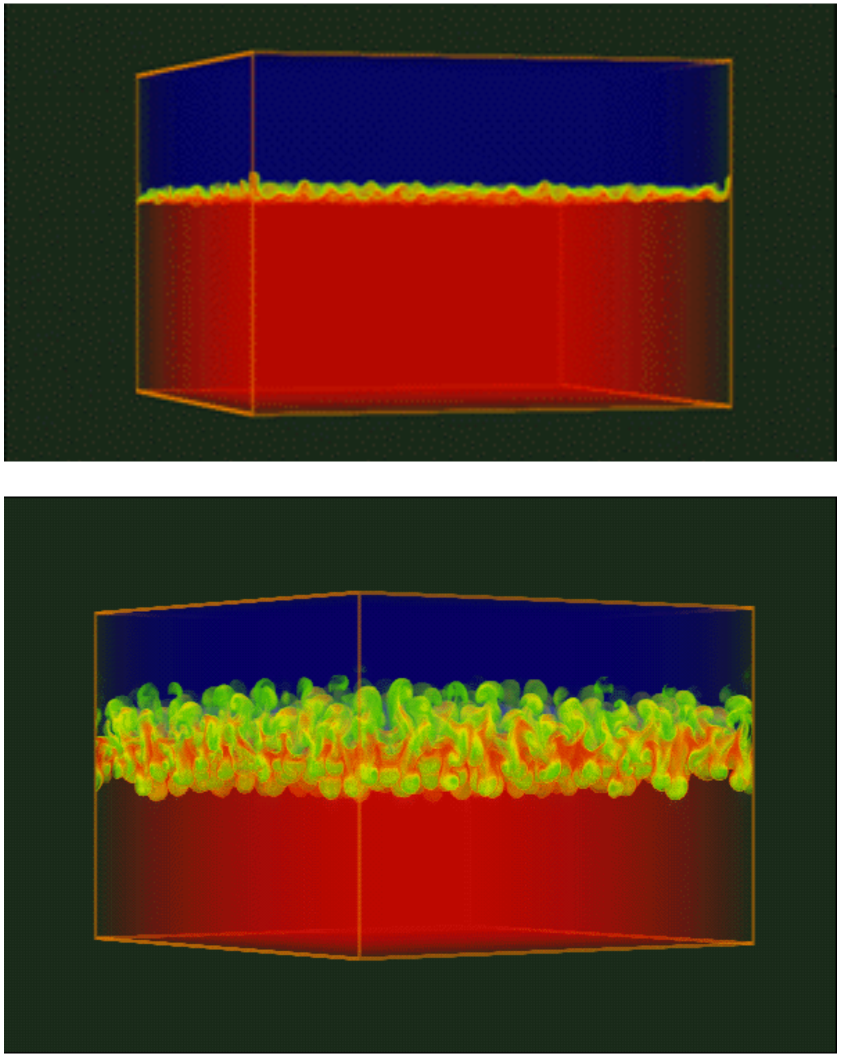
One obtains

$$\nabla p = \mathbf{J} \times \mathbf{B} + \rho \mathbf{g}$$

Ideal MHD Instabilities

- The primary sources of free energy in a static steady-state plasma is the plasma current density, and the plasma pressure gradient.
- Current-driven instabilities: kink modes
- Pressure-driven instabilities: Rayleigh-Taylor, interchange and/or ballooning modes

Rayleigh-Taylor Instabilities



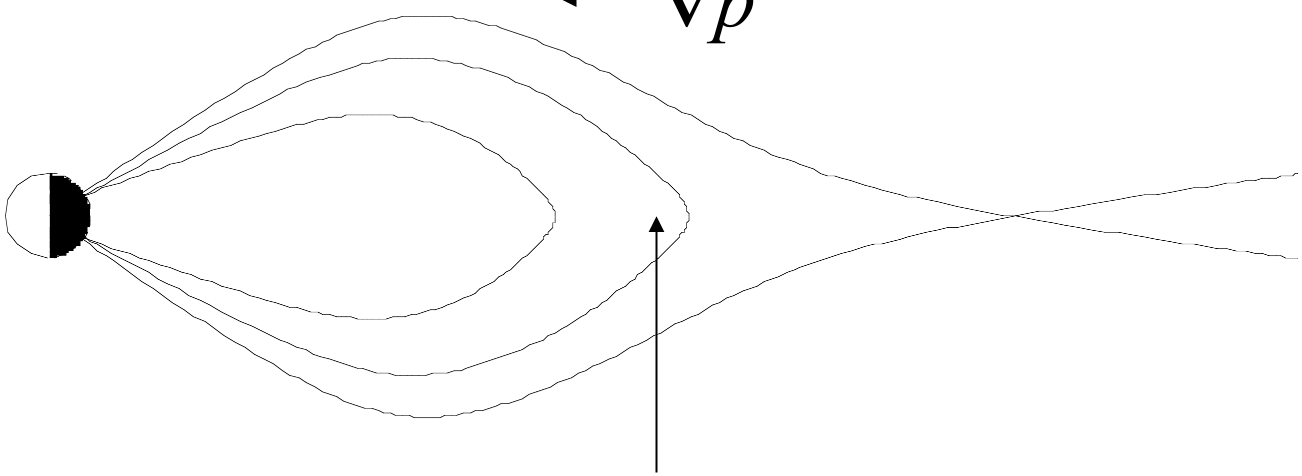
Heavy fluid resting on a lighter fluid in a gravitational field.

Is Near-Earth Magnetotail Ballooning Unstable?

$\leftarrow \ \vec{\kappa}$

"bad curvature"

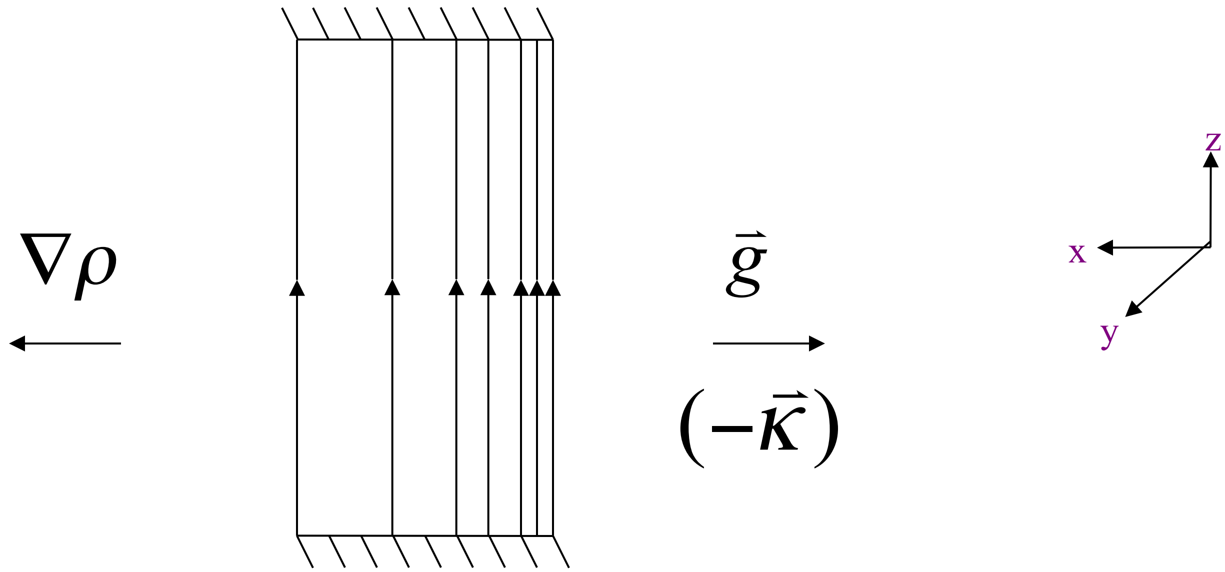
$\leftarrow \ \nabla p$



$\text{Near - Tail} \ (7 \sim 10 \ R_E)$

$\beta_{eq} \sim O(1) - O(100) \ !!$

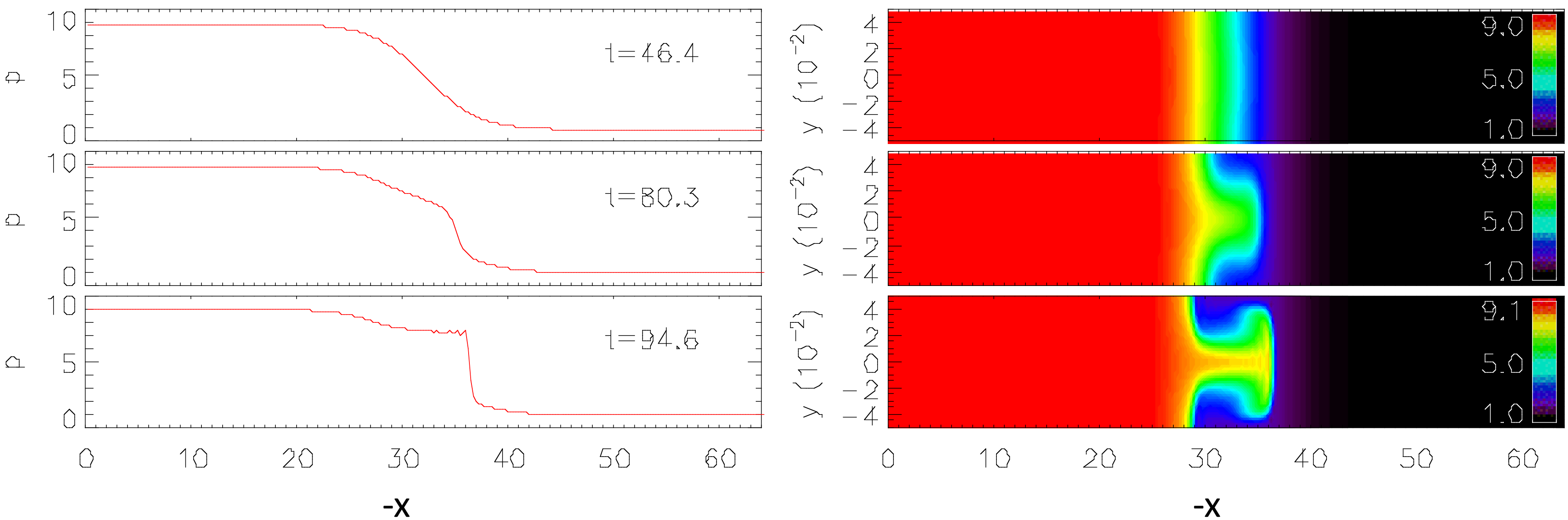
Box Model: 1D Near-Tail Equilibrium



$$\rho = \rho(x), \quad p = p(x), \quad \vec{B} = B(x)\hat{z}, \quad \vec{g} = -g\hat{x}$$

$$\frac{d\rho}{dx} > 0, \quad \frac{d}{dx}\left(p + \frac{B^2}{2}\right) = -\rho g$$

Nonlinear Rayleigh-Taylor or interchange instability: formation of shock-like coherent structures



(Zhu et al, 2005)

From *Magnetic Reconnection*
by E. Priest and T. Forbes

Frozen Flux/Field Theorem (Alfven's Theorem)

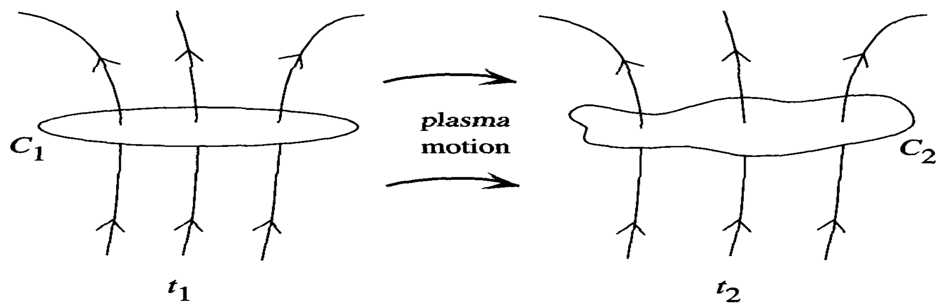


Fig. 1.6. Magnetic flux conservation: if a curve C_1 is distorted into C_2 by plasma motion, the flux through C_1 at t_1 equals the flux through C_2 at t_2.

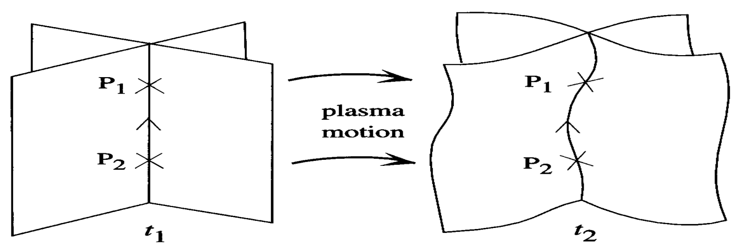


Fig. 1.7. Magnetic field-line conservation: if plasma elements P_1 and P_2 lie on a field line at time t_1, then they will lie on the same line at a later time t_2.

Magnetic Reconnection: Working Definition

If a plasma is perfectly conducting, that is, it obeys the ideal Ohm's law,

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0$$

B-lines are frozen in the plasma. Departures from ideal behavior, represented by

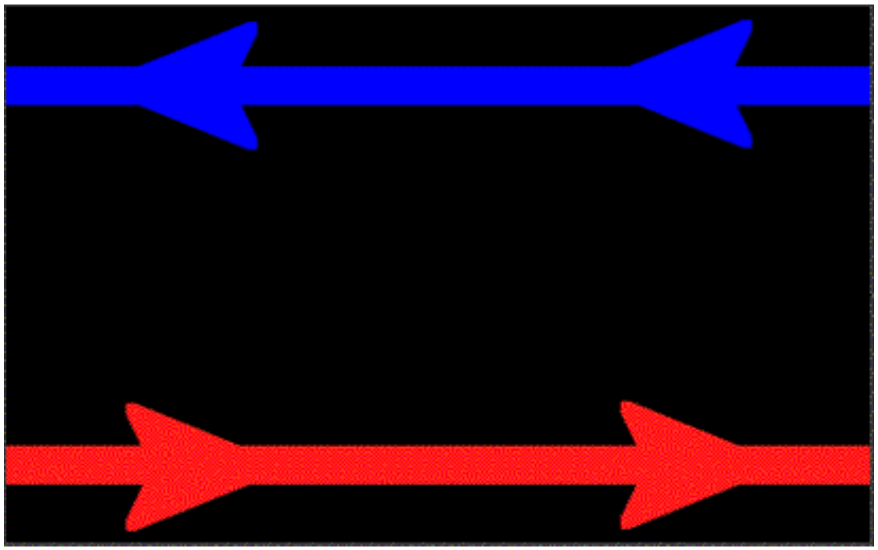
$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \mathbf{R}, \quad \nabla \times \mathbf{R} \neq \mathbf{0}$$

break ideal topological invariants, allowing field lines to break and reconnect.In generalized Ohm's law for collisionless plasmas, **R** contains resistivity, Hall current, electron inertia, and pressure.

Why is magnetic reconnection important?

Magnetic reconnection

- enables a system to access states of lower energy by topological relaxation of the magnetic field.
- energy liberated can be converted to flows, heating, and particle acceleration.



Lectures for which this material is directly pertinent

- *Dorelli* : The Earth's Magnetosphere
- *Donovan* : Substorms
- *Schriver*: The Solar Atmosphere
- *Judge* : The Chromosphere
- *Chandran:* Turbulence and Heating in the Solar Wind
- *Longcope*: Shocks
- *Sojka* : The Ionosphere

What is Turbulence?

- Webster's 1913 Dictionary: "*The quality or state of being turbulent; a disturbed state; tumult; disorder, agitation.*"
- Alexandre J. Chorin, Lectures on Turbulence Theory (Publish or Perish, Inc., Boston 1975): "*The distinguishing feature of turbulent flow is that its velocity field appears to be random and varies unpredictably. The flow does, however, satisfy a set of……equations, which are not random. This contrast is the source of much of what is interesting in turbulence theory.*"

Leonardo da Vinci (1500)

Navier-Stokes Equation: Fundamental Equation for Fluid Turbulence

$$\frac{\partial v}{\partial t} + v \cdot \nabla v = -\nabla p + R^{-1} \nabla^2 v + F$$

$$\nabla \cdot v = 0$$

R (= LV/viscosity) is called the Reynolds number.

Turbulence: Spatial Characteristics

- Turbulence couples large scales and small scales.
- The process of development of turbulence often starts out as large-scale motion by the excitation of waves of long wavelength that quickly produces waves of small wavelength by a domino effect.
- Wavelength and wavenumber

$$\lambda,\ k = 2\pi / \lambda$$

Clay Mathematics Institute

dedicated to increasing and disseminating mathematical knowledge

Millennium Prize Problems

P versus NP

The Hodge Conjecture

The Poincaré Conjecture

The Riemann Hypothesis

Yang–Mills Existence and Mass Gap

Navier–Stokes Existence and Smoothness

The Birch and Swinnerton–Dyer Conjecture

Announced 16:00, on Wednesday, May 24, 2000
Collège de France

Statement from the Directors and Scientific Advisory Board

Rules for the CMI Millennium Prize Problems

Historical Context

Press Statement

Press Reaction

Remarks

Information US: +1 617 868–8277 (Clay Mathematics Institute)

Hydrodynamic turbulence

Kolmogorov (1941): can get energy spectrum by dimensional analysis

Assumptions: ♦ isotropy

♦ local interaction in k-space

(energy moves from one k-shell to the next)

Energy cascade rate: $\varepsilon(k) \propto k^{\alpha} \underbrace{E(k)}^{\beta}$ = constant

Energy spectrum: $\int E(k)dk =$ total energy

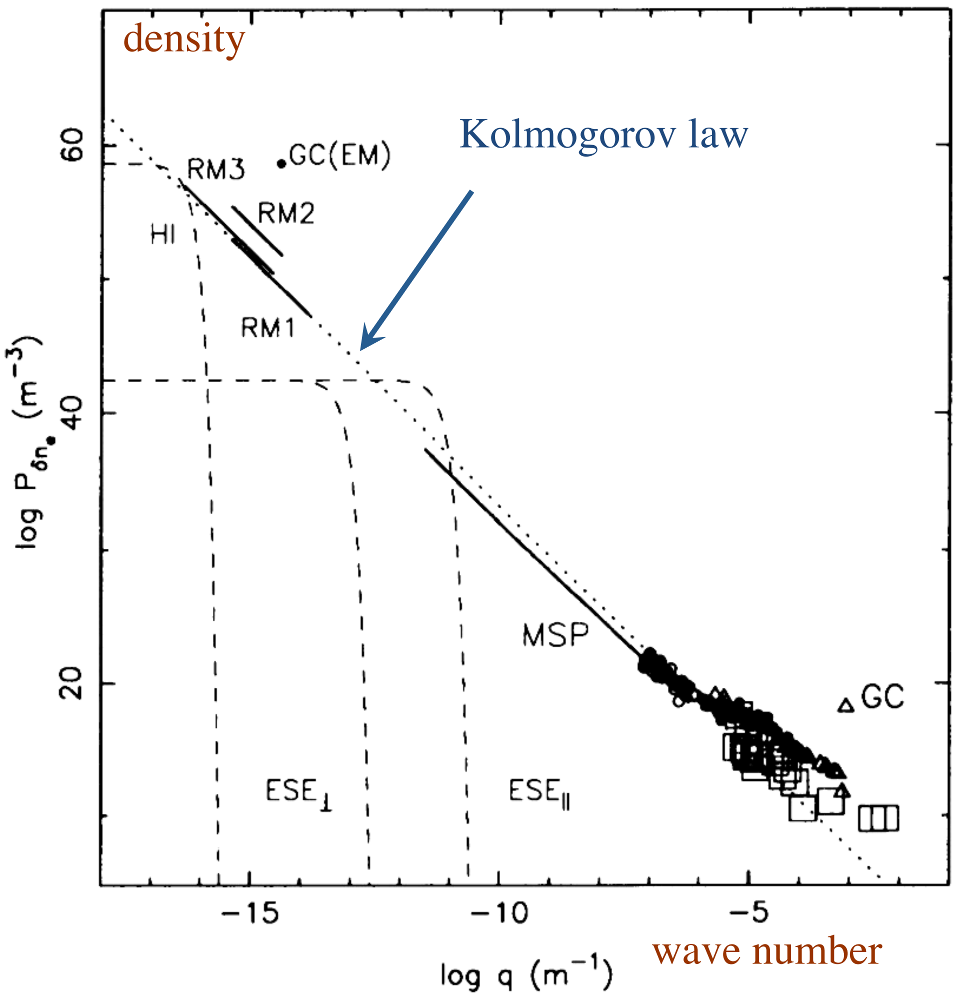
$\Rightarrow \ \alpha = 5/2, \quad \beta = 3/2$

Kolmogorov spectrum: $E(k) \sim C_{\mathrm{K}} \varepsilon^{2/3} k^{-5/3}$

Kolmogorov constant: $C_{\mathrm{K}} \sim 1.4 - 2$

Interstellar turbulence

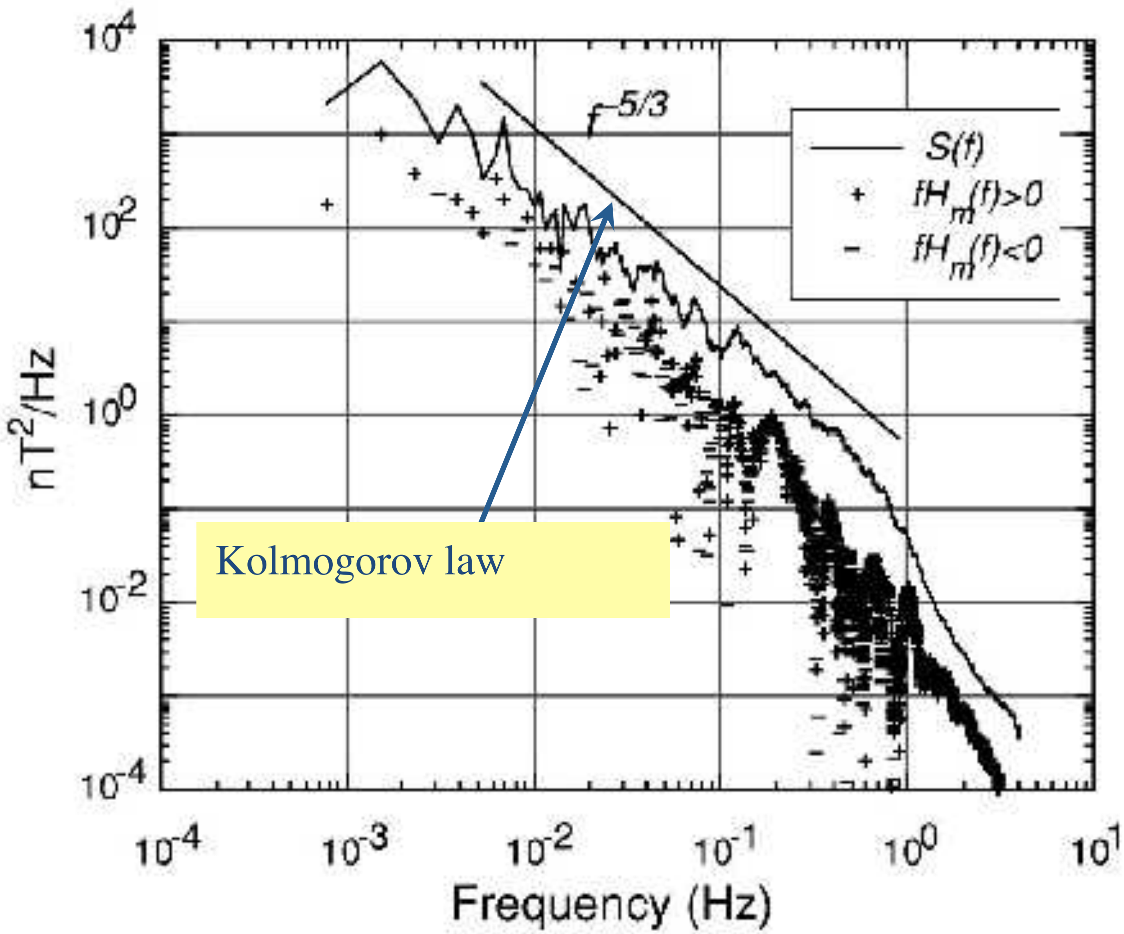
Observation:
power law relation between electron density spectrum and spatial scales



From Cordes (1999)

Solar wind turbulence

Observation: power law in magnetic energy spectrum



From Goldstein & Roberts (1999)

Much more on turbulence in….

- *Kasper*: The Solar Wind
- *Chandran*: Turbulence and Heating in the Solar Wind