Creation and destruction of magnetic fields

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Magnetic fields in the Universe

- Earth
  - Magnetic field present for $\sim 3.5 \cdot 10^9$ years, much longer than Ohmic decay time ($\sim 10^4$ years)
  - Strong variability on shorter time scales (10^3 years)
- Mercury, Ganymede, (Io), Jupiter, Saturn, Uranus, Neptune have large scale fields
- Sun
  - Magnetic fields from smallest observable scales to size of sun
  - 11 year cycle of large scale field (Movie)
  - Ohmic decay time $\sim 10^9$ years (in absence of turbulence)
- Other stars
  - Stars with outer convection zone: similar to sun
  - Stars with outer radiation zone: most likely primordial fields
- Galaxies
  - Field structure coupled to observed matter distribution (e.g. spirals)
  - Only dynamo that is directly observable

Scope of this lecture

- Processes of magnetic field generation and destruction in turbulent plasma flows
- Introduction to general concepts of dynamo theory (this is not a lecture about the solar dynamo!)
- Outline
  - MHD, induction equation
  - Some general remarks and definitions regarding dynamos
  - Large scale dynamos (mean field theory)
    - Kinematic theory
    - Characterization of possible dynamos
    - Non-kinematic effects
  - 3D simulations

MHD equations

The full set of MHD equations combines the induction equation with the Navier-Stokes equations including the Lorentz-force:

$$
\begin{aligned}
\frac{\partial \varrho}{\partial t} &= -\nabla \cdot (\varrho \mathbf{v}) \\
\varrho \frac{\partial \mathbf{v}}{\partial t} &= -\varrho (\mathbf{v} \cdot \nabla) \mathbf{v} - \nabla p + \varrho \mathbf{g} + \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B} + \nabla \cdot \boldsymbol{\tau} \\
\varrho \frac{\partial e}{\partial t} &= -\varrho (\mathbf{v} \cdot \nabla) e - p \nabla \cdot \mathbf{v} + \nabla \cdot (\kappa \nabla T) + Q_\nu + Q_\eta \\
\frac{\partial \mathbf{B}}{\partial t} &= \nabla \times (\mathbf{v} \times \mathbf{B} - \eta \nabla \times \mathbf{B})
\end{aligned}
$$

Assumptions:

- Validity of continuum approximation (enough particles to define averages)
- Non-relativistic motions, low frequencies
- Strong collisional coupling: validity of single fluid approximations, isotropic (scalar) gas pressure

MHD equations

Viscous stress tensor τ

$$\Lambda_{ik} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right)$$

$$\tau_{ik} = 2\varrho\nu\left(\Lambda_{ik} - \frac{1}{3}\delta_{ik}\nabla\cdot\mathbf{v}\right)$$

$$Q_\nu = \tau_{ik}\Lambda_{ik} \; ,$$

Ohmic dissipation Q_η

$$Q_\eta = \frac{\eta}{\mu_0}(\nabla\times\mathbf{B})^2 \; .$$

Equation of state

$$p = \frac{\varrho\, e}{\gamma - 1} \; .$$

ν, η and κ: viscosity, magnetic diffusivity and thermal conductivity
μ_0 denotes the permeability of vacuum

Kinematic approach

- Solving the 3D MHD equations is not always feasible
- Semi-analytical approach preferred for understanding fundamental properties of dynamos
- Evaluate turbulent induction effects based on induction equation for a given velocity field
  - Velocity field assumed to be given as 'background' turbulence, Lorentz-force feedback neglected (sufficiently weak magnetic field)
  - What correlations of a turbulent velocity field are required for dynamo (large scale) action?
  - Theory of onset of dynamo action, but not for non-linear saturation

- More detailed discussion of induction equation

Ohm's law

Equation of motion for drift velocity $\mathbf{v}_d$ of electrons

$$n_e m_e \left(\frac{\partial v_d}{\partial t} + \frac{v_d}{\tau_{ei}} \right) = n_e q_e (\mathbf{E} + \mathbf{v}_d \times \mathbf{B}) - \nabla p_e$$

τ_{ei}: collision time between electrons and ions
n_e: electron density
q_e: electron charge
m_e: electron mass
p_e: electron pressure
With the electric current: $\mathbf{j} = n_e\, q_e\, \mathbf{v}_d$ this gives the generalized Ohm's law:

$$\frac{\partial \mathbf{j}}{\partial t} + \frac{\mathbf{j}}{\tau_{ei}} = \frac{n_e q_e^2}{m_e}\mathbf{E} + \frac{q_e}{m_e}\mathbf{j} \times \mathbf{B} - \frac{q_e}{m_e}\nabla p_e$$

Simplifications:

- $\tau_{ei}\,\omega_L \ll 1$, $\omega_L = eB/m_e$: Larmor frequency
- neglect ∇p_e
- low frequencies (no plasma oscillations)

Ohm's law

Simplified Ohm's law

$$\mathbf{j} = \sigma \mathbf{E}$$

with the plasma conductivity

$$\sigma = \frac{\tau_{ei} n_e q_e^2}{m_e}$$

The Ohm's law we derived so far is only valid in the co-moving frame of the plasma. Under the assumption of non-relativistic motions this transforms in the laboratory frame to

$$\mathbf{j} = \sigma \left(\mathbf{E} + \mathbf{v} \times \mathbf{B} \right)$$

Induction equation*

Using Ampere's law $\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$ yields for the electric field in the laboratory frame

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B} + \frac{1}{\mu_0 \sigma} \nabla \times \mathbf{B}$$

leading to the induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} = \nabla \times (\mathbf{v} \times \mathbf{B} - \eta \nabla \times \mathbf{B})$$

with the magnetic diffusivity

$$\eta = \frac{1}{\mu_0 \sigma} \ .$$

Advection, diffusion, magnetic Reynolds number

L: typical length scale U: typical velocity scale L/U: time unit

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left(\mathbf{v} \times \mathbf{B} - \frac{1}{R_m} \nabla \times \mathbf{B} \right)$$

with the magnetic Reynolds number

$$R_m = \frac{U\,L}{\eta} \; .$$

$R_m \ll 1$: diffusion dominated regime

$$\frac{\partial \mathbf{B}}{\partial t} = \eta \Delta \mathbf{B} \; .$$

Only decaying solutions with decay (diffusion) time scale

$$\tau_{\mathrm{d}} \sim \frac{L^2}{\eta}$$

Advection, diffusion, magnetic Reynolds number

$R_m \gg 1$ advection dominated regime (ideal MHD)

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B})$$

Equivalent expression

$$\frac{\partial \mathbf{B}}{\partial t} = -(\mathbf{v} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{v} - \mathbf{B}\, \nabla \cdot \mathbf{v}$$

- advection of magnetic field
- amplification by shear (stretching of field lines)
- amplification through compression

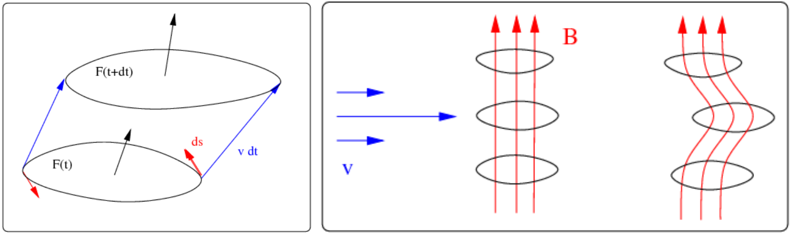
Advection, diffusion, magnetic Reynolds number

Object	$\eta[\mathrm{m}^2/\mathrm{s}]$	$L[\mathrm{m}]$	$U[\mathrm{m/s}]$	R_m	τ_d
earth (outer core)	2	10^6	10^{-3}	300	10^4 years
sun (plasma conductivity)	1	10^8	100	10^{10}	10^9 years
sun (turbulent conductivity)	10^8	10^8	100	100	3 years
liquid sodium lab experiment	0.1	1	10	100	10 s

Alfven's theorem

Let Φ be the magnetic flux through a surface F with the property that its boundary ∂F is moving with the fluid:

$$\Phi = \int_F \mathbf{B} \cdot d\mathbf{f} \longrightarrow \frac{d\Phi}{dt} = 0$$



- Flux is 'frozen' into the fluid
- Field lines 'move' with plasma

Dynamos: Motivation

- For $\mathbf{v} = 0$ magnetic field decays on timescale $\tau_d \sim L^2/\eta$
- Earth and other planets:
  - Evidence for magnetic field on earth for $3.5 \cdot 10^9$ years while $\tau_d \sim 10^4$ years
  - Permanent rock magnetism not possible since $T > T_{\rm Curie}$ and field highly variable $\longrightarrow$ field must be maintained by active process
- Sun and other stars:
  - Evidence for solar magnetic field for $\sim 300\,000$ years (^{10}Be)
  - Most solar-like stars show magnetic activity independent of age
  - Indirect evidence for stellar magnetic fields over life time of stars
  - But $\tau_d \sim 10^9$ years!
  - Primordial field could have survived in radiative interior of sun, but convection zone has much shorter diffusion time scale ~ 10 years (turbulent diffusivity)

Mathematical definition of dynamo

S bounded volume with the surface ∂S, $\mathbf{B}$ maintained by currents contained within S, $B \sim r^{-3}$ asymptotically,

$$\begin{aligned} \frac{\partial \mathbf{B}}{\partial t} &= \nabla \times (\mathbf{v} \times \mathbf{B} - \eta \nabla \times \mathbf{B}) && \text{in } S \\ \nabla \times \mathbf{B} &= 0 && \text{outside } S \\ [\mathbf{B}] &= 0 && \text{across } \partial S \\ \nabla \cdot \mathbf{B} &= 0 \end{aligned}$$

$\mathbf{v} = 0$ outside S, $\mathbf{n} \cdot \mathbf{v} = 0$ on ∂S and

$$E_{\mathrm{kin}} = \int_S \frac{1}{2} \varrho \mathbf{v}^2 \, dV \leq E_{\mathrm{max}} \quad \forall\, t$$

$\mathbf{v}$ is a dynamo if an initial condition $\mathbf{B} = \mathbf{B}_0$ exists so that

$$E_{\mathrm{mag}} = \int_{-\infty}^{\infty} \frac{1}{2\mu_0} \mathbf{B}^2 \, dV \geq E_{\mathrm{min}} \quad \forall\, t$$

Decompose the magnetic field into large scale part and small scale part (energy carrying scale of turbulence) $\mathbf{B} = \overline{\mathbf{B}} + \mathbf{B}'$:

$$E_{\rm mag} = \int \frac{1}{2\mu_0} \overline{\mathbf{B}}^2 \, dV + \int \frac{1}{2\mu_0} \overline{\mathbf{B}'^2} \, dV \ .$$

- Small scale dynamo: $\overline{\mathbf{B}}^2 \ll \overline{\mathbf{B}'^2}$
- Large scale dynamo: $\overline{\mathbf{B}}^2 \geq \overline{\mathbf{B}'^2}$

Almost all turbulent (chaotic) velocity fields are small scale dynamos for sufficiently large R_m, large scale dynamos require additional large scale symmetries (see second half of this lecture)

What means large/small in practice (Sun)?

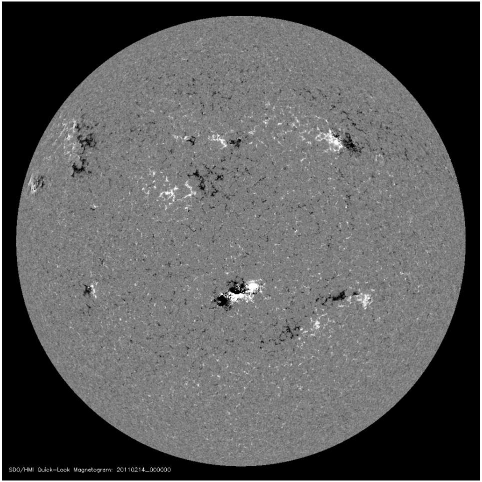


Figure: Full disk magnetogram SDO/HMI

What means large/small in practice (Sun)?

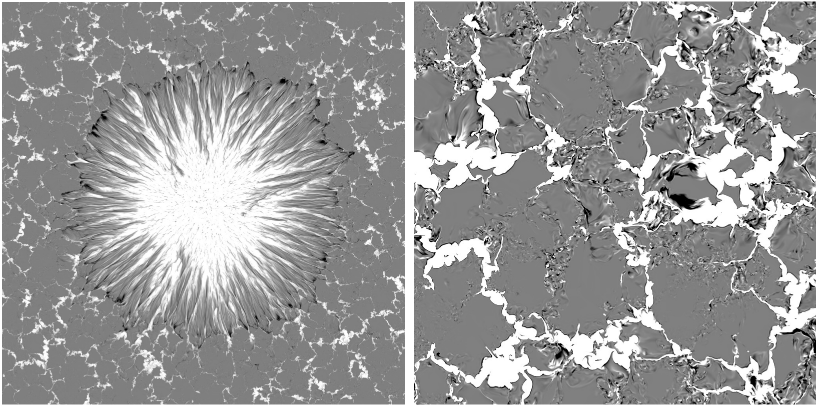
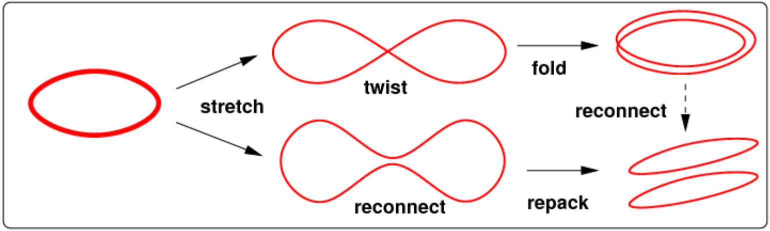


Figure: Numerical sunspot simulation. Dimensions: Left 50x50 Mm, Right: 12.5x12.5 Mm

Large scale/small scale dynamos



- Amplification through field line stretching
- Twist-fold required to repack field into original volume
- Magnetic diffusivity allows for change of topology

Slow/fast dynamos

Influence of magnetic diffusivity on growth rate

- Fast dynamo: growth rate independent of R_m (stretch-twist-fold mechanism)
- Slow dynamo: growth rate limited by resistivity (stretch-reconnect-repack)

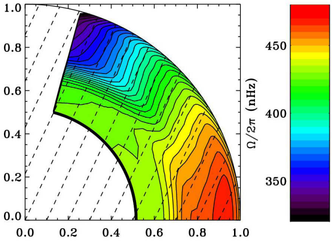
- Fast dynamos relevant for most astrophysical objects since $R_m \gg 1$
- Dynamos including (resistive) reconnection steps can be fast provided the reconnection is fast

Differential rotation and meridional flow

Induction effects of axisymmetric flows on axisymmetric field:

$$\mathbf{B} = B\mathbf{e}_\Phi + \nabla \times (A\mathbf{e}_\Phi)$$
$$\mathbf{v} = v_r\mathbf{e}_r + v_\theta\mathbf{e}_\theta + \Omega\, r \sin\theta\,\mathbf{e}_\Phi$$

Differential rotation most dominant shear flow in stellar convection zones:



Meridional flow by-product of DR, observed as poleward surface flow in case of the sun

Differential rotation and meridional flow

Spherical geometry:

$$\frac{\partial B}{\partial t} + \frac{1}{r}\left(\frac{\partial}{\partial r}(r v_r B) + \frac{\partial}{\partial \theta}(v_\theta B)\right) =$$

$$r \sin \mathbf{B}_p \cdot \nabla\Omega + \eta\left(\Delta - \frac{1}{(r\sin\theta)^2}\right) B$$

$$\frac{\partial A}{\partial t} + \frac{1}{r\sin\theta}\mathbf{v}_p \cdot \nabla(r\sin\theta A) = \eta\left(\Delta - \frac{1}{(r\sin\theta)^2}\right) A$$

- Meridional flow: Independent advection of poloidal and toroidal field
- Differential rotation: Source for toroidal field (if poloidal field not zero)
- Diffusion: Sink for poloidal and toroidal field
- No term capable of maintaining poloidal field against Ohmic decay!

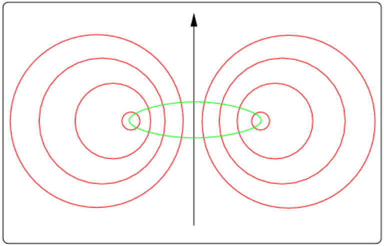
Differential rotation and meridional flow

- Weak poloidal seed field can lead to significant field amplification
- No source term for poloidal field
- Decay of poloidal field on resistive time scale
- Ultimate decay of toroidal field
- Not a dynamo!
- What is needed?

- Source for poloidal field

Cowling's anti-dynamo theorem

A stationary axisymmetric magnetic field with currents limited to a finite volume in space cannot be maintained by a velocity field with finite amplitude.



Ohm's law of the form $\mathbf{j} = \sigma \mathbf{E}$ only decaying solutions, focus here on $\mathbf{j} = \sigma(\mathbf{v} \times \mathbf{B})$.
On O-type neutral line $\mathbf{B}_p$ is zero, but $\mu_0 \mathbf{j}_t = \nabla \times \mathbf{B}_p$ has finite value, but cannot be maintained by $(\mathbf{v} \times \mathbf{B})_t = (\mathbf{v}_p \times \mathbf{B}_p)$.

Some history:

- 1919 Sir Joeseph Larmor: Solar magnetic field maintained by motions of conducting fluid?
- 1937 Cowling's anti-dynamo theorem and many others
- 1955 Parker: decomposition of field in axisymmetric and non-axisymmetric parts, average over induction effects of non-axisymmetric field
- 1964 Braginskii, Steenbeck, Krause: Mathematical frame work of mean field theory developed
- last 2 decades 3D dynamo simulations

Reynolds rules

We need to define an averaging procedure to define the mean and the fluctuating field.

For any function f and g decomposed as $f = \overline{f} + f'$ and $g = \overline{g} + g'$ we require that the Reynolds rules apply

$$
\begin{aligned}
\overline{\overline{f}} &= \overline{f} \longrightarrow \overline{f'} = 0 \\
\overline{f+g} &= \overline{f} + \overline{g} \\
\overline{f\overline{g}} &= \overline{f}\overline{g} \longrightarrow \overline{f'\overline{g}} = 0 \\
\overline{\partial f/\partial x_i} &= \partial \overline{f}/\partial x_i \\
\overline{\partial f/\partial t} &= \partial \overline{f}/\partial t \,.
\end{aligned}
$$

Examples:

- Longitudinal average (mean = axisymmetric component)
- Ensemble average (mean = average over several realizations of chaotic system)

Meanfield induction equation

Average of induction equation:

$$\frac{\partial \overline{\mathbf{B}}}{\partial t} = \nabla \times \left(\overline{\mathbf{v}' \times \mathbf{B}'} + \overline{\mathbf{v}} \times \overline{\mathbf{B}} - \eta \nabla \times \overline{\mathbf{B}} \right)$$

New term resulting from small scale effects:

$$\overline{\mathcal{E}} = \overline{\mathbf{v}' \times \mathbf{B}'}$$

Fluctuating part of induction equation:

$$\left(\frac{\partial}{\partial t} - \eta \Delta \right) \mathbf{B}' - \nabla \times (\overline{\mathbf{v}} \times \mathbf{B}') = \nabla \times \left(\mathbf{v}' \times \overline{\mathbf{B}} + \mathbf{v}' \times \mathbf{B}' - \overline{\mathbf{v}' \times \mathbf{B}'} \right)$$

Kinematic approach: $\mathbf{v}'$ assumed to be given

- Solve for $\mathbf{B}'$, compute $\overline{\mathbf{v}' \times \mathbf{B}'}$ and solve for $\overline{\mathbf{B}}$
- Term $\mathbf{v}' \times \mathbf{B}' - \overline{\mathbf{v}' \times \mathbf{B}'}$ leading to higher order correlations (closure problem)

Mean field expansion of turbulent induction effects

Exact expressions for $\overline{\boldsymbol{\mathcal{E}}}$ exist only under strong simplifying assumptions (see homework assignment).

In general $\overline{\boldsymbol{\mathcal{E}}}$ is a linear functional of $\overline{\mathbf{B}}$:

$$\overline{\mathcal{E}}_i(\mathbf{x},t) = \int_{-\infty}^{\infty} d^3x' \int_{-\infty}^{t} dt' \, \mathcal{K}_{ij}(\mathbf{x},t,\mathbf{x}',t') \, \overline{B}_j(\mathbf{x}',t') \; .$$

Can be simplified if a sufficient scale separation is present:

- $l_c \ll L$
- $\tau_c \ll \tau_L$

Leading terms of expansion:

$$\overline{\mathcal{E}}_i = a_{ij}\overline{B}_j + b_{ijk}\frac{\partial \overline{B}_j}{\partial x_k}$$

In stellar convection zones scale separation also only marginally justified (continuous turbulence spectrum)!

Symmetry constraints

Decomposing a_{ij} and $\partial \overline{B}_j / \partial x_k$ into symmetric and antisymmetric components:

$$a_{ij} = \underbrace{\frac{1}{2}(a_{ij} + a_{ji})}_{\alpha_{ij}} + \underbrace{\frac{1}{2}(a_{ij} - a_{ji})}_{-\varepsilon_{ijk}\gamma_k}$$

$$\frac{\partial \overline{B}_j}{\partial x_k} = \frac{1}{2}\left(\frac{\partial \overline{B}_j}{\partial x_k} + \frac{\partial \overline{B}_k}{\partial x_j}\right) + \underbrace{\frac{1}{2}\left(\frac{\partial \overline{B}_j}{\partial x_k} - \frac{\partial \overline{B}_k}{\partial x_j}\right)}_{-\frac{1}{2}\varepsilon_{jkl}(\nabla \times \overline{\mathbf{B}})_l}$$

Leads to:

$$\overline{\mathcal{E}}_i = \alpha_{ik}\overline{B}_k + \varepsilon_{ijk}\gamma_j\overline{B}_k - \underbrace{\frac{1}{2} b_{ijk}\varepsilon_{jkl}}_{\beta_{il} - \varepsilon_{ilm}\delta_m}(\nabla \times \overline{\mathbf{B}})_l + \ldots$$

Symmetry constraints

Overall result:

$$\overline{\boldsymbol{\mathcal{E}}} = \boldsymbol{\alpha}\overline{\mathbf{B}} + \boldsymbol{\gamma} \times \overline{\mathbf{B}} - \boldsymbol{\beta}\,\boldsymbol{\nabla} \times \overline{\mathbf{B}} - \boldsymbol{\delta} \times (\boldsymbol{\nabla} \times \overline{\mathbf{B}}) + \ldots$$

With:

$$\alpha_{ij} = \frac{1}{2}\left(a_{ij} + a_{ji}\right)\,, \qquad \gamma_i = -\frac{1}{2}\varepsilon_{ijk}a_{jk}$$

$$\beta_{ij} = \frac{1}{4}\left(\varepsilon_{ikl}b_{jkl} + \varepsilon_{jkl}b_{ikl}\right)\,, \qquad \delta_i = \frac{1}{4}\left(b_{jji} - b_{jij}\right)$$

Symmetry constraints

α, β, γ and δ depend on large scale symmetries of the system defining the symmetry properties of the turbulence (e.g. rotation and stratification). Additional to that the expansion

$$\overline{\boldsymbol{\mathcal{E}}} = \boldsymbol{\alpha}\overline{\mathbf{B}} + \boldsymbol{\gamma} \times \overline{\mathbf{B}} - \boldsymbol{\beta}\,\boldsymbol{\nabla} \times \overline{\mathbf{B}} - \boldsymbol{\delta} \times \boldsymbol{\nabla} \times \overline{\mathbf{B}} + \ldots$$

is a relation between polar and axial vectors:

- $\overline{\boldsymbol{\mathcal{E}}}$: polar vector, independent from handedness of coordinate system
- $\mathbf{B}$: axial vector, involves handedness of coordinate system in definition (curl operator, cross product)

Handedness of coordinate system pure convention (contains no physics), consistency requires:

- $\boldsymbol{\alpha}$, $\boldsymbol{\delta}$: pseudo tensor
- $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$: true tensors

Symmetry constraints

Turbulence with rotation and stratification

- true tensors: δ_{ij}, g_i, $g_i g_j$, $\Omega_i\Omega_j$, $\Omega_i\varepsilon_{ijk}$
- pseudo tensors: ε_{ijk}, Ω_i, $\Omega_i g_j$, $g_i\varepsilon_{ijk}$

Symmetry constraints allow only certain combinations:

$$\alpha_{ij} = \alpha_0(\mathbf{g}\cdot\boldsymbol{\Omega})\delta_{ij} + \alpha_1\left(g_i\Omega_j + g_j\Omega_i\right)\ , \qquad \gamma_i = \gamma_0 g_i + \gamma_1\varepsilon_{ijk}g_j\Omega_k$$

$$\beta_{ij} = \beta_0\,\delta_{ij} + \beta_1\,g_i g_j + \beta_2\,\Omega_i\Omega_j\ , \qquad \delta_i = \delta_0\Omega_i$$

The scalars $\alpha_0 \ldots \delta_0$ depend on quantities of the turbulence such as rms velocity and correlation times scale.

- isotropic turbulence: only $\boldsymbol{\beta}$
- $+$ stratification: $\boldsymbol{\beta} + \boldsymbol{\gamma}$
- $+$ rotation: $\boldsymbol{\beta} + \boldsymbol{\delta}$
- $+$ stratification $+$ rotation: $\boldsymbol{\alpha}$ can exist

Simplified expressions

Assuming $|\mathbf{B}'| \ll |\overline{\mathbf{B}}|$ in derivation + additional simplification for (quasi) isotropic, non-mirror symmetric, (weakly) inhomogeneous turbulence (see homework assignment):

$$\overline{v_i' v_j'} \sim \delta_{ij},\ \alpha_{ij} = \alpha\delta_{ij},\ \beta_{ij} = \eta_t\delta_{ij}$$

Leads to:

$$\frac{\partial \overline{\mathbf{B}}}{\partial t} = \nabla \times \left[\alpha\overline{\mathbf{B}} + (\overline{\mathbf{v}} + \gamma) \times \overline{\mathbf{B}} - (\eta + \eta_t)\nabla \times \overline{\mathbf{B}}\right]$$

with the scalar quantities

$$\alpha = -\frac{1}{3}\tau_c\,\overline{\mathbf{v}' \cdot (\nabla \times \mathbf{v}')}, \qquad \eta_t = \frac{1}{3}\tau_c\,\overline{\mathbf{v}'^2}$$

and vector

$$\gamma = -\frac{1}{6}\tau_c\nabla\overline{\mathbf{v}'^2} = -\frac{1}{2}\nabla\eta_t$$

Expressions are independent of η (in this approximation), indicating fast dynamo action!

Turbulent diffusivity - destruction of magnetic field

Turbulent diffusivity dominant dissipation process for large scale field in case of large R_m:

$$\eta_t = \frac{1}{3}\tau_c \, \overline{\mathbf{v}'^2} \sim L \, v_{\rm rms} \sim R_m \eta \gg \eta$$

- Formally η_t comes from advection term (transport term, non-dissipative)
- Turbulent cascade transporting magnetic energy from the large scale L to the micro scale l_m (advection + reconnection)

$$\eta \mathbf{j}_m^2 \sim \eta_t \bar{\mathbf{j}}^2 \longrightarrow \frac{B_m}{l_m} \sim \sqrt{R_m}\frac{\overline{B}}{L}$$

Important: The large scale determines the energy dissipation rate, l adjusts to allow for the dissipation on the microscale.

Present for isotropic homogeneous turbulence

Turbulent diamagnetism, turbulent pumping

Expulsion of flux from regions with larger turbulence intensity 'diamagnetism'

$$\gamma = -\frac{1}{2}\boldsymbol{\nabla}\eta_t$$

Turbulent pumping (stratified convection):

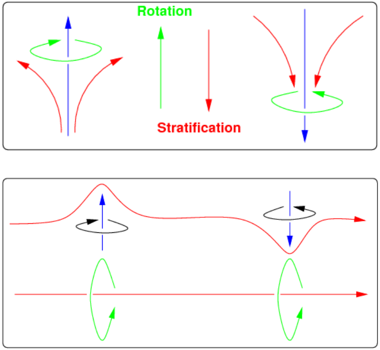
$$\gamma = -\frac{1}{6}\tau_c\boldsymbol{\nabla}\overline{\mathbf{v}'^2}$$

- Upflows expand, downflows converge
- Stronger velocity and smaller filling factor of downflows
- Mean induction effect of up- and downflow regions does not cancel
- Downward transport found in numerical simulations

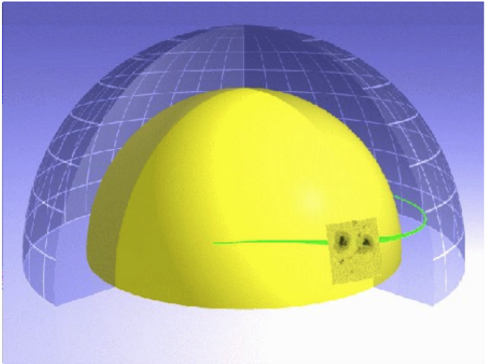
Requires inhomogeneity (stratification)

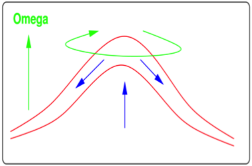
$$\alpha = -\frac{1}{3}\tau_c \overline{\mathbf{v}' \cdot (\nabla \times \mathbf{v}')} \qquad H_k = \overline{\mathbf{v}' \cdot (\nabla \times \mathbf{v}')} \text{ kinetic helicity}$$

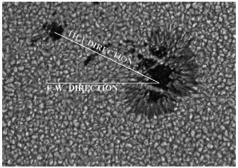
Requires rotation + additional preferred direction (stratification)



Babcock-Leighton α-effect







- Similar to kinetic α-effect, but driven by magnetic buoyancy
- Leading polarities have larger propability to reconnect across equator with counterpart on other hemisphere
- Polarity of hemisphere = polarity of following sunspots

What is needed to circumvent Cowling's theorem?

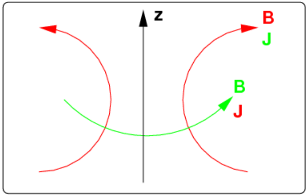
- Crucial for Cowling's theorem: Impossibility to drive a current parallel to magnetic field
- Cowling's theorem does not apply to mean field if a mean current can flow parallel to the mean field (since total field non-axisymmetric this is not a contradiction!)

$$\bar{\mathbf{j}} = \tilde{\boldsymbol{\sigma}} \left(\overline{E} + \bar{\mathbf{v}} \times \overline{\mathbf{B}} + \boldsymbol{\gamma} \times \overline{\mathbf{B}} + \boldsymbol{\alpha} \overline{\mathbf{B}} \right)$$

$\tilde{\boldsymbol{\sigma}}$ contains contributions from η, β and δ.
Ways to circumvent Cowling:

- α-effect
- anisotropic conductivity (off diagonal elements + δ-effect)

α^2-dynamo



Induction of field parallel to current (producing helical field!)

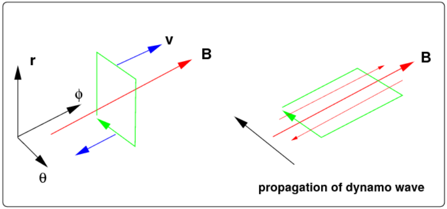
$$\frac{\partial \overline{\mathbf{B}}}{\partial t} = \nabla \times (\alpha \overline{\mathbf{B}}) = \alpha \mu_0 \overline{\mathbf{j}}$$

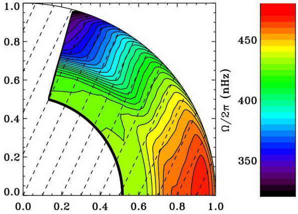
Dynamo cycle:

$$\mathbf{B}_t \xrightarrow{\alpha} \mathbf{B}_p \xrightarrow{\alpha} \mathbf{B}_t$$

- Poloidal and toroidal field of similar strength
- In general stationary solutions

$\alpha\Omega$-, $\alpha^2\Omega$-dynamo





Dynamo cycle:

$$\mathbf{B}_t \xrightarrow{\alpha} \mathbf{B}_p \xrightarrow{\Omega} \mathbf{B}_t$$

- Toroidal field much stronger that poloidal field
- In general traveling (along lines of constant Ω) and periodic solutions

$\alpha\Omega$-dynamo

$$\frac{\partial B}{\partial t} + \frac{1}{r}\left(\frac{\partial}{\partial r}(r v_r B) + \frac{\partial}{\partial \theta}(v_\theta B)\right) = r \sin \mathbf{B}_p \cdot \nabla\Omega$$
$$+ \eta\left(\Delta - \frac{1}{(r\sin\theta)^2}\right) B$$
$$\frac{\partial A}{\partial t} + \frac{1}{r\sin\theta}\mathbf{v}_p \cdot \nabla(r\sin\theta A) = \alpha B + \eta\left(\Delta - \frac{1}{(r\sin\theta)^2}\right) A$$

- Dimensionless measure for strength of Ω- and α-effect

$$D_\Omega = \frac{R^2\Delta\Omega}{\eta_t} \qquad D_\alpha = \frac{R\alpha}{\eta_t}$$

- Dynamo excited if dynamo number

$$D = D_\Omega D_\alpha > D_{crit}$$

$\alpha\Omega$-dynamo without meridional flow

$$\frac{\partial B}{\partial t} = r \sin \mathbf{B}_p \cdot \nabla\Omega + \eta\left(\Delta - \frac{1}{(r\sin\theta)^2}\right) B$$

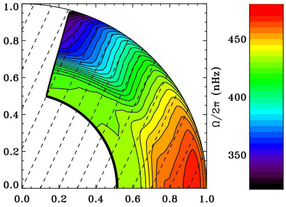
$$\frac{\partial A}{\partial t} = \alpha B + \eta\left(\Delta - \frac{1}{(r\sin\theta)^2}\right) A$$

- Cyclic behavior:

$$P \propto (\alpha|\nabla\Omega|)^{-1/2}$$

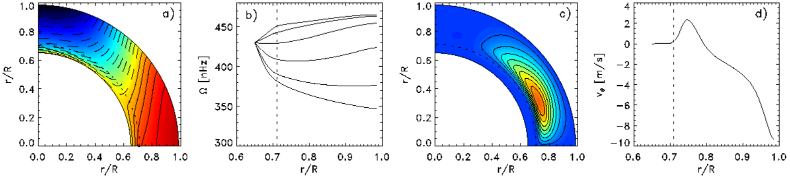
- Propagation of magnetic field along contourlines of Ω "dynamo-wave"
- Direction of propagation "Parker-Yoshimura-Rule":

$$\mathbf{s} = \alpha\nabla\Omega \times \mathbf{e}_\phi$$



Movie: $\alpha\Omega$-dynamo

$\alpha\Omega$-dynamo with meridional flow



Meridional flow:

- Poleward at top of convection zone
- Equatorward at bottom of convection zone

Effect of advection:

- Equatorward propagation of activity
- Correct phase relation between poloidal and toroidal field
- Circulation time scale of flow sets dynamo period
- Requirement: Sufficiently low turbulent diffusivity

Movie: Flux-transport-dynamo (M. Dikpati, HAO)

Dynamos and magnetic helicity

Magnetic helicity (integral measure of field topology):

$$H_m = \int \mathbf{A} \cdot \mathbf{B} \, dV$$

has following conservation law (no helicity fluxes across boundaries):

$$\frac{d}{dt} \int \mathbf{A} \cdot \mathbf{B} \, dV = -2\mu_0 \, \eta \int \mathbf{j} \cdot \mathbf{B} \, dV$$

Decomposition into small and large scale part:

$$\begin{aligned}
\frac{d}{dt} \int \overline{\mathbf{A}} \cdot \overline{\mathbf{B}} \, dV &= +2 \int \overline{\mathcal{E}} \cdot \overline{\mathbf{B}} \, dV - 2\mu_0 \, \eta \int \overline{\mathbf{j}} \cdot \overline{\mathbf{B}} \, dV \\
\frac{d}{dt} \int \overline{\mathbf{A}' \cdot \mathbf{B}'} \, dV &= -2 \int \overline{\mathcal{E}} \cdot \overline{\mathbf{B}} \, dV - 2\mu_0 \, \eta \int \overline{\mathbf{j}' \cdot \mathbf{B}'} \, dV
\end{aligned}$$

Dynamos and magnetic helicity

Dynamos have helical fields:

- α-effect induces magnetic helicity of same sign on large scale
- α-effect induces magnetic helicity of opposite sign on small scale

Asymptotic staturation (time scale $\sim R_m \tau_c$):

$$\overline{\mathbf{j}' \cdot \mathbf{B}'} = -\bar{\mathbf{j}} \cdot \overline{\mathbf{B}} \longrightarrow \frac{|\overline{B}|}{|B'|} \sim \sqrt{\frac{L}{l_c}}$$

$$\overline{\mathbf{j}' \cdot \mathbf{B}'} = -\frac{\alpha \overline{\mathbf{B}}^2}{\mu_0 \eta} + \frac{\eta_t}{\eta} \bar{\mathbf{j}} \cdot \overline{\mathbf{B}}$$

Time scales:

- Galaxy: $\sim 10^{25}$ years ($R_m \sim 10^{18}$, $\tau_c \sim 10^7$ years)
- Sun: $\sim 10^8$ years
- Earth: $\sim 10^6$ years

Non-kinematic effects

Proper way to treat them: 3D simulations

- Still very challenging
- Has been successful for geodynamo, but not for solar dynamo

Semi-analytical treatment of Lorentz-force feedback in mean field models:

- Macroscopic feedback: Change of the mean flow (differential rotation, meridional flow) through the mean Lorentz-force

$$\bar{\mathbf{f}} = \bar{\mathbf{j}} \times \overline{\mathbf{B}} + \overline{\mathbf{j}' \times \mathbf{B}'}$$

- Mean field model including mean field representation of full MHD equations:
  Movie: Non-kinematic flux-transport dynamo
- Microscopic feedback: Change of turbulent induction effects (e.g. α-quenching)

Microscopic feedback

Feedback of Lorentz force on small scale motions:

- Intensity of turbulent motions significantly reduced if $\frac{1}{2\mu_0}B^2 > \frac{1}{2}\varrho v_{rms}^2$. Typical expression used

$$\alpha = \frac{\alpha_k}{1 + \frac{\overline{\mathbf{B}}^2}{B_{eq}^2}}$$

  with the equipartition field strength $B_{eq} = \sqrt{\mu_0 \varrho} v_{rms}$
- Similar quenching also expected for turbulent diffusivity
- Additional quenching of α due to topological constraints possible (helicity conservation)
  Controversial !

Microscopic feedback

Symmetry of momentum and induction equation $\mathbf{v}' \leftrightarrow \mathbf{B}'/\sqrt{\mu_0\varrho}$:

$$\frac{d\mathbf{v}'}{dt} = \frac{1}{\mu_0\varrho}(\overline{\mathbf{B}}\cdot\nabla)\mathbf{B}' + \ldots$$

$$\frac{d\mathbf{B}'}{dt} = (\overline{\mathbf{B}}\cdot\nabla)\mathbf{v}' + \ldots$$

$$\overline{\boldsymbol{\mathcal{E}}} = \overline{\mathbf{v}'\times\mathbf{B}'}$$

Strongly motivates magnetic term for α-effect (Pouquet et al. 1976):

$$\alpha = \frac{1}{3}\tau_c\left(\frac{1}{\varrho}\overline{\mathbf{j}'\cdot\mathbf{B}'} - \overline{\boldsymbol{\omega}'\cdot\mathbf{v}'}\right)$$

- Kinetic α: $\overline{\mathbf{B}} + \mathbf{v}' \longrightarrow \mathbf{B}' \longrightarrow \overline{\boldsymbol{\mathcal{E}}}$
- Magnetic α: $\overline{\mathbf{B}} + \mathbf{B}' \longrightarrow \mathbf{v}' \longrightarrow \overline{\boldsymbol{\mathcal{E}}}$

Microscopic feedback

From helicity conservation one expects

$$\overline{\mathbf{j}' \cdot \mathbf{B}'} \sim -\alpha \overline{\mathbf{B}}^2$$

leading to algebraic quenching

$$\alpha = \frac{\alpha_k}{1 + g\,\frac{\overline{\mathbf{B}}^2}{B_{eq}^2}}$$

With the asymptotic expression (steady state)

$$\overline{\mathbf{j}' \cdot \mathbf{B}'} = -\frac{\alpha \overline{\mathbf{B}}^2}{\mu_0 \eta} + \frac{\eta_t}{\eta} \bar{\mathbf{j}} \cdot \overline{\mathbf{B}}$$

we get

$$\alpha = \frac{\alpha_{\mathrm{k}} + \frac{\eta_t^2}{\eta} \frac{\mu_0 \bar{\mathbf{j}} \cdot \overline{\mathbf{B}}}{B_{\mathrm{eq}}^2}}{1 + \frac{\eta_t}{\eta} \frac{\overline{\mathbf{B}}^2}{B_{\mathrm{eq}}^2}}$$

Microscopic feedback

Catastrophic α-quenching ($R_m \gg 1!$) in case of steady state and homogeneous $\overline{\mathbf{B}}$:

$$\alpha = \frac{\alpha_{\mathrm{k}}}{1 + R_m \frac{\overline{\mathbf{B}}^2}{B_{\mathrm{eq}}^2}}$$

If $\bar{\mathbf{j}} \cdot \overline{\mathbf{B}} \neq 0$ (dynamo generated field) and η_t unquenched:

$$\alpha \approx \eta_t \, \mu_0 \frac{\bar{\mathbf{j}} \cdot \overline{\mathbf{B}}}{\overline{\mathbf{B}}^2} \sim \frac{\eta_t}{L} \sim \frac{\eta_t}{l_c} \frac{l_c}{L} \sim \alpha_k \frac{l_c}{L}$$

- In general α-quenching dynamic process: linked to time evolution of helicity
- Boundary conditions matter: Loss of small scale current helicity can alleviate catastrophic quenching
- Catastrophic α-quenching turns large scale dynamo into slow dynamo

3D simulations

Why not just solving the full system to account for all non-linear effects?

- Most systems have $R_e \gg R_m \gg 1$, requiring high resolution
- Large scale dynamos evolve on time scales $\tau_c \ll t \ll \tau_\eta$, requiring long runs compared to convective turn over
- 3D simulations successful for geodynamo
  - $R_m \sim 300$: all relevant magnetic scales resolvable
  - Incompressible system
- Solar dynamo: Ingredients can be simulated
  - Compressible system: density changes by 10^6 through convection zone
  - Boundary layer effects: Tachocline, difficult to simulate (strongly subadiabatic stratification, large time scales)
  - Magnetic structures down to 1000 km most likely important Evolve 5000^3 box over 1000 τ_c!
  - Small scale dynamos can be simulated (for $P_m \sim 1$)

Summarizing remarks

Destruction of magnetic field:

- Turbulent diffusivity: cascade of magnetic energy from large scale to dissipation scale (advection+reconnection)
- Enhances dissipation of large field by a factor R_m

Creation of magnetic field:

- Small scale dynamo (non-helical)
  - Amplification of field at and below energy carrying scale of turbulence
  - Stretch-twist-fold-(reconnect)
  - Produces non-helical field and does not require helical motions
  - Current research: behavior for $P_m \ll 1$
- Large scale dynamo (helical)
  - Amplification of field on scales larger than scale of turbulence
  - Produces helical field and does require helical motions
  - Requires rotation + additional symmetry direction (controversial $\Omega \times J$ effect does not require helical motions)
  - Current research: catastrophic vs. non-catastrophic quenching