Problem Set Exercises: Dynamos in Planets and Stars

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The MHD Approximation

In the lecture presentation, you are told that the MHD approximation involves approximations to Maxwell's equations when fluid velocities are $<<$ speed of light. In this section, you will derive these approximations starting from the equations on the slide entitled "2. Dynamo Theory Basics".

1. **The Ampere-Maxwell Law**: Start with the differential forms of Faraday's law and the Ampere-Maxwell law. Take B to be a characteristic magnetic field scale, E to be a characteristic electric field scale, L to be a characteristic length scale and τ a characteristic time scale.

   (a) Using order of magnitude arguments, use Faraday's law to estimate E in terms of B, L, and τ.

   (b) Use your expression for E to estimate the size of the displacement current term in the Ampere-Maxwell equation in terms of B, L, τ and c (the speed of light).

   (c) Now compare this estimate for the displacement current to an estimate for the left hand side of the Ampere-Maxwell law. From this, show that the displacement current can be neglected in the Ampere-Maxwell law if $L/\tau = u << c$.

2. **The Lorentz transformations**: The Lorentz transformations between reference frames are given by:

$$\gamma = \frac{1}{\sqrt{1-u^2/c^2}} \qquad \vec{x}'_{\parallel} = \gamma(\vec{x}_{\parallel} - \vec{u}t) \qquad \vec{x}'_{\perp} = \vec{x}_{\perp}$$

$$\vec{E}'_{\parallel} = \vec{E}_{\parallel} \qquad \vec{E}'_{\perp} = \gamma\left[\vec{E}_{\perp} + \vec{u}\times\vec{B}\right]$$

$$\vec{B}'_{\parallel} = \vec{B}_{\parallel} \qquad \vec{B}'_{\perp} = \gamma\left[\vec{B}_{\perp} - \frac{1}{c^2}\vec{u}\times\vec{E}\right]$$

$$\vec{J}'_{\parallel} = \gamma\left[\vec{J}_{\parallel} + \rho_e\vec{u}\right] \qquad \vec{J}'_{\perp} = \vec{J}_{\perp} \qquad \rho'_e = \rho_e - \frac{1}{c^2}\vec{u}\cdot\vec{J}$$

   In the MHD limit, $\gamma \approx 1$ since $u << c$, simplifying the transformations. You will want to use this in all the transformations above. Other simplifications can be made using order of magnitude estimates:

   (a) Consider the transformation for the perpendicular component of the magnetic field. Using your estimate for E in question 1, show that in the MHD limit, the second term (the one with u and E) can be neglected relative to the first term. This result, in combination with the transformation for the parallel component of $\vec{B}$, results in the MHD magnetic field transformation: $\vec{B}' = \vec{B}$.

   (b) Now consider the transformation for the parallel component of the current density. Use Gauss' law to estimate $\rho_e\vec{u}$ in terms of ϵ_0, B, u, τ. Then use the MHD version of the Ampere-Maxwell law you derived in question 1 to estimate the current density in terms of B, μ_0 and L. Compare the sized of these 2 terms in the MHD limit to demonstrate that $\vec{J}' = \vec{J}$.

(c) Finally, combine the parallel and perpendicular components of the electric field to demonstrate that $\vec{E}' = \vec{E} + \vec{u} \times \vec{B}$.

3. **The Lorentz force**: Using expressions you derived above, demonstrate that the electric force is much smaller than the magnetic force in the Lorentz force equation in the MHD limit resulting in: $F_L = \vec{J} \times \vec{B}$.

Magnetic Induction Equation: Alfven's Theorem Example

The lecture notes discussed Alfven's Theorem. An example was mentioned where we consider a perfect conductor with a constant vertical magnetic field: ($\vec{B} = B_0\hat{z}$) and a simple horizontal shear flow ($\vec{u} = \frac{Uz}{L}\hat{x}$) where z is the vertical coordinate. Here we will work through the steps of solving the MIE in the perfectly conducting limit to demonstrate that the generated field is given by: $\vec{B} = \frac{UtB_0}{L}\hat{x} + B_0\hat{z}$

1. Plug in the above expressions for $\vec{B}_0$ and $\vec{u}$ into the MIE for a perfect conductor.
2. Solve the resulting ODE for the generated field $\vec{B}$.
3. You might think you are done, but there is one more important step. You've just found that a new component of the field (in the $\hat{x}$ direction) has been generated. Its possible that the original velocity field will now act on your newly generated field to generate even newer field! So now plug in your expression for $\vec{B}$ into your original ODE and show that for this example, no newer component of the field is generated. (*Note: For examples with more complicated flows and initial fields, this iterative process can go on indefinitely which is why there aren't tons of analytic solutions for MIE examples, even in the absence of the diffusion term.*)

Mean Field Equations

The lecture notes introduced mean field dynamo models. Here we will derive the mean field equations from the magnetic induction equation. We begin by writing the magnetic field in terms of its mean and perturbation components:

$$\vec{B}(\vec{r},t) = \vec{B}_0(\vec{r},t) + \vec{b}(\vec{r},t)$$

where:

$$\vec{B}_0(\vec{r},t) = <\vec{B}(\vec{r},t)>$$

$$<\vec{b}(\vec{r},t)> = 0$$

We will also assume the velocity field has no mean component and hence is given by:

$$\vec{U}(\vec{r},t) = \vec{u}(\vec{r},t)$$

$$<\vec{u}(\vec{r},t)> = 0$$

We will also assume it is incompressible:

$\nabla \cdot \vec{u} = 0$

If we plug these expression into the magnetic induction equation we get:

$$\frac{\partial(\vec{B}_0 + \vec{b})}{\partial t} = \nabla \times [\vec{u} \times (\vec{B}_0 + \vec{b})] + \lambda \nabla^2 (\vec{B}_0 + \vec{b}) \tag{1}$$

As discussed in the lecture, the mean is defined as:

$$< f >= \frac{3}{4\pi a^3} \int_a f dV \tag{2}$$

Here are some useful properties about mean fields that we will use to derive the mean field equations. f and g are physical quantities and c and d are real numbers.:

- $< f >= f$ if f is constant in space
- $< cf + dg >= c < f > +d < g >$
- $\partial_t < f >=< \partial_t f >$
- $\nabla < f >=< \nabla f >$ as well as other operations with ∇
- $<< f > g >=< f >< g >$
- $<< f >>=< f >$

Ok, on to the exercise...

1. Take the mean of equation (1) and use the mean field properties above to derive the equation for the mean field $\vec{B}_0$:

$$\frac{\partial \vec{B}_0}{\partial t} = \nabla \times (< \vec{u} \times \vec{b} >) + \lambda \nabla^2 \vec{B}_0 \tag{3}$$

2. Now subtract this equation from equation (1) to drive an equation for the perturbation field $\vec{b}$:

$$\frac{\partial \vec{b}}{\partial t} = \nabla \times (\vec{u} \times \vec{B}_0) + \nabla \times (\vec{u} \times \vec{b} - < \vec{u} \times \vec{b} >) + \lambda \nabla^2 \vec{b} \tag{4}$$

Assuming u is given, this equation demonstrates that there is a LINEAR relation between $\vec{b}$ and $\vec{B}_0$. This means that $\varepsilon =< \vec{u} \times \vec{b} >$ will also depend linearly on $\vec{B}_0$.

3. If we assume $\vec{B}_0$ is uniform in space and constant in time, then we must have $\varepsilon_i = \alpha_{ij} B_{0j}$ The pseudotensor α_{ij} is determined in principle by the properties of $\vec{u}$ and by λ. This is the simplest mean field approximation. If instead, we let $\vec{B}_0$ be slowly varying in space, then we can expand ε as a Taylor expansion that rapidly converges. In this case, we can write:

$$\varepsilon_i = \alpha_{ij} B_{0j} + \beta_{ijk} \frac{\partial B_{0j}}{\partial x_k} + \gamma_{ijkl} \frac{\partial^2 B_{0j}}{\partial x_k \partial x_l} + ... \tag{5}$$

This was the equation given in the notes. (admittedly, you didn't have to do anything for this question).

4. Now we can make assumptions about the properties of the small scale motions and fields to get simpler forms for the tensors α_{ij} and β_{ijk} If the turbulence is homogeneous (the same at every point), then α_{ij} is independent of position. If the turbulence is isotropic (the same in every direction) then α_{ij}, β_{ijk} are isotropic and can be written as:

$$\alpha_{ij} = \alpha\delta_{ij} \qquad \beta_{ijk} = \beta\epsilon_{ijk} \tag{6}$$

Here α is a pseudoscalar and β is a pure scalar. Plug these expressions into the first 2 terms of equation (5) to show that the emf can be written:

$$\vec{\varepsilon} = \alpha\vec{B}_0 - \beta(\nabla \times \vec{B}_0) + ... \tag{7}$$

BONUS MATERIAL:

5. Starting from Ohm's law, find an expression for the mean current density in terms of α and β under the same assumptions as in question 4.

6. Then take the mean of Ampere's law and plug it into your expression to derive the following equation for the mean current density:

$$< \vec{J} >= \sigma_{eff}\left(< \vec{E} > +\alpha\vec{B}_0\right) \Rightarrow \tag{8}$$

where

$$\sigma_{eff} = \sigma / \left(1 + \frac{\sigma\beta}{\mu_0}\right) \tag{9}$$

This demonstrates that one effect of small-scale turbulent motions is to DECREASE the effective conductivity (or equivalently, INCREASE the effective magnetic diffusivity).

7. Finally, plug in your expression for ε given in equation (16) into equation (8) to derive a common form of the mean field equation:

$$\frac{\partial\vec{B}_0}{\partial t} = \alpha\nabla \times \vec{B}_0 + (\lambda + \beta)\nabla^2\vec{B}_0 \tag{10}$$

Here is how to interpret this equation: The mean field can grow through the induction term that involves the effects of small scale motions (turbulence) on the mean field. This is known as the "α effect. Turbulence can also enhance diffusivity through the β effect (as we saw already).