PROBLEM SET

Heliophysics Summer School

July, 2013

Problem Set for Shocks and Particle Acceleration

There is probably only time to attempt one or two of these questions.

In the tutorial session discussion will concentrate on questions 2 and 3, but move onto other questions if there is time.

SECTION Problems

1. The upstream deHoffmann-Teller velocity is given by

$$\mathbf{V}_{HT} = -\frac{\hat{\mathbf{n}} \times (\mathbf{B} \times \mathbf{V}_{un})}{\hat{\mathbf{n}} \cdot \mathbf{B}}$$

Show that this is also the de Hoffmann-Teller velocity in the region downstream, i.e., that automatically the downstream flow is field-aligned when transforming into the upstream de Hoffmann-Teller frame.

2. Derive the following expression for the ratio of downstream to upstream tangential magnetic field component through a MHD discontinuity

$$\frac{B_{dt}}{B_{ut}} = r\frac{{v_{un}}^2 - {c_{A,u}}^2}{{v_{un}}^2 - r\ {c_{A,u}}^2}$$

where $r = \frac{v_{dn}}{v_{un}} = \frac{\rho_d}{\rho_u}$ is the compression ratio and $c_{A,u} = B_{un}/(\mu_0 \rho_u)^{1/2}$ the upstream Alfvén (intermediate) speed. Use for the derivation the tangential momentum jump condition and the condition that the tangential electric field is constant through the shock.

3. A proton (e.g. some energetic charged particle) is trapped between 2 moving magnetic mirrors in which the field strength increases by a factor of 5. It has an initial kinetic energy $W = 1$ keV and $v_\perp = v_\parallel$ in the midplane between the two mirrors. Each mirror is moving towards this mid-plane with velocity $V_m = 10$ km/s. Initially the mirrors are separated by a distance $L = 10^{10}$km.

(a) Using the invariance of particle magnetic moment $\mu_m = W_\perp/B$ find the energy to which the proton will be accelerated before it escapes the system.

(b) How long will it take to reach that energy?

Hints:

(a) Using conservation of magnetic moment, the minimum field required to mirror a particle of initial pitch angle α_0 and field B_0 is $B_m = B_0/\sin^2\alpha_0$.

(b) Treat the mirrors as flat pistons and show that $v_\parallel$ increases by $2V_m$ at each bounce.

(c) Compute the number of bounces needed

(d) Accuracy to a factor of 2 is sufficient.

[This question continues overleaf . . .]

4. For the exactly perpendicular shock, derive in the high Mach limit, where $M_A \gg 1$ and $M_{cs} \gg 1$, the following relation for the compression ratio r,

$$r = \frac{\gamma + 1}{\gamma - 1}.$$

Show that it follows for $\gamma = 5/3$ that the maximum compression ratio is $r = 4$.

Method: Take the normal momentum conservation equation and write it out explicitly in terms of upstream and downstream quantities. Rewrite in terms of r using $\rho_d = r\rho_u$ and $B_d = r\,B_u$. Do a similar exercise for the energy jump condition. From these two equations one can eliminate P_d, leaving an equation for r.

Note the definitions for the Alfvén and sonic Mach numbers:

$$M_A = \frac{V_u}{B_u/\sqrt{\mu_0 \rho_u}} \quad (1)$$

$$M_{cs} = \frac{V_u}{\sqrt{\gamma\, P_u/\rho_u}}. \quad (2)$$

5. Consider a one-dimensional, steady state, shock at which the x axis is taken parallel to the unit shock normal vector $\hat{\mathbf{n}}$, such that all quantities depend only on x. The jump across the shock for any quantity X is written as $[X] = X_u - X_d$, where the subscripts u and d refer to the upstream and downstream values, respectively. For a time-steady, one-dimensional shock, the condition $\partial X/\partial x = 0$ implies $[X] = 0$.

The magnetic field and velocity vectors can be split into components normal and parallel to the shock surface. For example,

$$\mathbf{B} = B_x \hat{\mathbf{n}} + \mathbf{B}_t,$$

where $\mathbf{B}_t$ is the transverse component vector, and B_x is the normal component.

(a) Using Maxwell's equations and the one-fluid MHD equations, show that

$$[B_x] = 0$$
$$[\rho V_x] = 0$$

(b) From the tranverse component of the MHD momentum equation (without gravity), and using the results above, deduce the jump condition:

$$\left[\rho V_x \mathbf{V}_t - \frac{B_x}{\mu_0}\mathbf{B}_t\right] = 0.$$

(c) A further jump condition is

$$[V_x \mathbf{B}_t - B_x \mathbf{V}_t] = 0.$$

[This question continues overleaf . . .]

(You need NOT prove this.)

Using these jump conditions show that the upstream and downstream transverse components of the magnetic field are parallel, i.e., that $\mathbf{B}_{dt} = \alpha \mathbf{B}_{ut}$ where α is a non-zero scalar. Hence, show that

$$\hat{\mathbf{n}} \cdot (\mathbf{B}_u \times \mathbf{B}_d) = 0$$

i.e., that the three vectors $\hat{\mathbf{n}}$, $\mathbf{B}_u$ and $\mathbf{B}_d$ are coplanar.

(d) From the previous result, and the further result (which you do not need to prove)

$$\hat{\mathbf{n}} \cdot (\mathbf{B}_u - \mathbf{B}_d) = 0$$

explain why the vector

$$\mathbf{N} = (\mathbf{B}_u \times \mathbf{B}_d) \times (\mathbf{B}_u - \mathbf{B}_d)$$

can be used to determine the shock normal from the measured upstream and magnetic field vectors.

6. Consider a fluid with negligible pressure which transports nonrelativistic energetic particles which are coupled to it by a constant diffusion coefficient κ. The fluid equations and the Parker convection-diffusion equation for the particles are:

$$\frac{\partial \rho}{\partial t} = \nabla \cdot (\rho \mathbf{V}) = 0 \quad (3)$$

$$\rho \frac{\partial V}{\partial t} + \rho (\mathbf{V} \cdot \nabla) \mathbf{V} = -\nabla P \quad (4)$$

$$\frac{\partial P}{\partial t} + \mathbf{V} \cdot \nabla P - \kappa \nabla^2 P + \gamma (\nabla \cdot \mathbf{V}) P = 0 \quad (5)$$

where P is the pressure of the energetic particles and $\gamma = 5/3$.

(a) Consider a stationary planar system with variations in the x direction only and rewrite the equations for this system.

(b) Find three integrals of the system (mass flux, momentum flux, and energy flux conservation). To do this rewrite PdV/dx appearing in one term as $d/dx(PV) - VdP/dx$. In the terms involving dP/dx use the simplified version of the momentum equation to replace dP/dx by the term involving V and dV/dx. The resulting equation can be integrated easily.

(c) Determine the three constants by setting $V = V_0 > 0$, $\rho = \rho_0$, and $P = 0$ as $x \to -\infty$.

(d) Derive the following equation for $V(x)$ alone by eliminating P in the energy flux integral:

$$\frac{2\kappa}{\gamma + 1} \frac{dV}{dx} = (V - V_0) \left(V - \frac{\gamma - 1}{\gamma + 1} V_0 \right).$$

 [This question continues overleaf ...]

(e) Solve this equation for $V(x)$ and interpret the constant of integration. Derive expressions for $\rho(x)$ and $P(x)$ and plot them schematically. What does the solution represent?

7. Consider a perpendicular, time-steady shock, in a frame in which the shock is stationary in the plane $x = 0$, with an upstream flow velocity $\mathbf{V} = -V\hat{\mathbf{x}}$, and a uniform upstream magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$. The normal to the shock surface points upstream $\hat{\mathbf{n}} = \hat{\mathbf{x}}$, so that the upstream region has $x > 0$.

(a) Assuming ideal MHD, what is the upstream electric field $\mathbf{E}$? Give your answer in component form.

(b) A particle of mass m and charge q hits the shock with exactly the upstream flow velocity, and is then reflected specularly, i.e., it reverses its component of velocity normal to the shock.

From the particle's equation of motion, obtain an analytic solution for the velocity $\mathbf{u}$ and position $\mathbf{x}$ in component form, assuming an initial position of $(0, 0, 0)$. Assume that the magnetic and electric fields are uniform throughout the particle's motion. Use the definition $\Omega = qB/m$.

(c) Describe briefly with a sketch the particle's motion after reflection. Derive the following expression for the maximum distance that a reflected particle reaches upstream:

$$x_{\text{max}} = \frac{V}{\Omega}\left(\sqrt{3} - \frac{\pi}{3}\right)$$

(d) Describe briefly the overall magnetic field structure of a high Mach number perpendicular shock as observed in space. What is the role of reflected ions at such shocks, and how do they influence the structure?

[Next section overleaf]

USEFUL INFORMATION

(i) The Lorentz force on a particle of charge q moving in electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ respectively is given by

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

(ii) Maxwell's Equations

$$\begin{aligned}
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \times \mathbf{B} &= \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \\
\nabla \cdot \mathbf{E} &= \frac{\rho_q}{\epsilon_0} \\
\nabla \cdot \mathbf{B} &= 0
\end{aligned}$$

where $\mu_0 \epsilon_0 = 1/c^2$.

(iv) The MHD equations for a plasma with electrical conductivity σ:

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) &= 0 \\
\rho \left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \mathbf{V} &= -\nabla p + \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B} \\
\left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \left(p \rho^{-\gamma} \right) &= 0 \\
\frac{\partial \mathbf{B}}{\partial t} &= \nabla \times (\mathbf{V} \times \mathbf{B}) + \frac{1}{\mu_0 \sigma} \nabla^2 \mathbf{B} \\
\mathbf{E} + \mathbf{V} \times \mathbf{B} &= \mathbf{j}/\sigma
\end{aligned}$$

(v) Conservation forms of the ideal MHD momentum and energy equations

$$\begin{aligned}
\frac{\partial (\rho \mathbf{V})}{\partial t} + \nabla \cdot \left[\rho \mathbf{V}\mathbf{V} + \left(p + \frac{B^2}{2\mu_0} \right) \boldsymbol{I} - \frac{\mathbf{B}\mathbf{B}}{\mu_0} \right] &= \mathbf{0} \\
\frac{\partial}{\partial t} \left(\frac{1}{2}\rho V^2 + \frac{p}{\gamma - 1} + \frac{B^2}{2\mu_0} \right) + \nabla \cdot \left(\mathbf{V}\frac{1}{2}\rho V^2 + \frac{\gamma}{\gamma - 1} p \mathbf{V} + \frac{\mathbf{E} \times \mathbf{B}}{\mu_0} \right) &= 0
\end{aligned}$$

(vi) The following vector identities and relations

$$\begin{aligned}
\nabla \times (\mathbf{a} \times \mathbf{b}) &= \mathbf{a}(\nabla \cdot \mathbf{b}) + (\mathbf{b} \cdot \nabla)\mathbf{a} - \mathbf{b}(\nabla \cdot \mathbf{a}) - (\mathbf{a} \cdot \nabla)\mathbf{b} \\
(\nabla \times \mathbf{B}) \times \mathbf{B} &= (\mathbf{B} \cdot \nabla)\mathbf{B} - \nabla \left(\frac{B^2}{2} \right) \\
\nabla \times (\nabla \times \mathbf{B}) &= \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} \\
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \\
(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} &= (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}
\end{aligned}$$