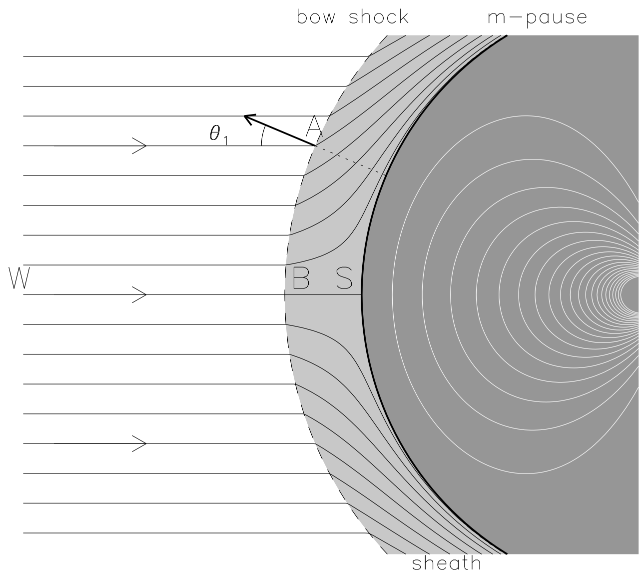
Shocks: Homework

1. Solar wind with Mach number $M_{sw} \gg 1$ hits the Earth's magnetosphere. Assume the magnetic field in the wind is small enough to be neglected.



A simplified of the day-side magnetosphere. Flow streamlines are shown as solid curves originating at the Sun, far to the left. The bow shock is a dashed arc and the magnetopause is a thick solid arc. The magnetosphere proper is dark grey, with some white magnetic field lines shown. The magnetosheath is the lighter greay region between the bow shock and the magnetopause.

   a. Consider the central streamline, W–B–S, where the flow passes normal to the bow shock at point B and then comes to rest at stagnation point S on the magnetopause. Show that the temperature, T_B, at the point B is some multiple of v_{sw}^2, where v_{sw} is the solar wind speed. Assume a a fully ionized hydrogen plasma.

   b. Assume the flow in the sheath is adiabatic and thus satisfies Bernoulli's equation

$$\tfrac{1}{2}v^2 \ + \ 5\frac{k_\mathrm{B}T}{m_p} \ = \ \mathrm{const.} \quad .$$

   Find the temperature T_S at the stagnation point. What is T_S when $v_{sw} = 800$ km/s?

   c. By what factors are the plasma density at points B and S enhanced above the density at point W? Continue to assume $M_{sw} \gg 1$.

   d. Now consider a point away from the central line, such as A, where the shock normal makes an angle $\theta_1 \ll 1$ with the incoming flow. In the limit $M_{sw} \gg 1$ show that the flow is deflected by an angle $\Delta\theta \simeq 3\theta_1$.

   e. For what incidence angles θ_1 is the post-shock speed, including normal and tangential components, greater than the local sound speed? i.e. supersonic.

   f. The results from part c. suggest the flow in the sheath, at least near the central line $\theta \ll 1$, is roughly incompressible (what is its Mach number?). An incompressible flow can be written using a stream function $\mathbf{v} = \nabla\psi \times \hat{\boldsymbol{\phi}}/r\sin\theta$. The particular stream function[1]

$$\psi(r,\theta) \ = \ A\left(\frac{r^4}{R_{mp}^4} - \frac{R_{mp}}{r}\right)\sin^2\theta \quad ,$$

[1] This particular stream function also matches the normal flow and vorticity at the shock, but you don't need to show this. (Lighthill, *J. Fluid Mech.* **2**, 1–32 [1957]).

vanishes at $r = R_{mp}$, making it a flow around a *spherical magnetopause*. Assume the bow shock is a spherical shell concentric with the magnetopause. Use the results from d. to find its radius.

2. In a solar flare, rapid energy release heats a coronal plasma and drives material downward along field lines. Because the flow is parallel to the field we can consider this a simple *hydrodynamic shock* driving material at speed v_f into a coronal material with sound speed c_{s0}. You may assume $v_f \gg c_{s0}$ and use only the hypersonic jump conditions. This shock then encounters the *transition region* which we can take to be a simple *contact discontinuity* across which the density increases by a factor $r \gg 1$. We wish to understand the effect of this interaction and we will do so *ignoring thermal conduction.*[2]

   a. What is the speed of the initial downward shock relative to the contact discontinuity?

   b. First consider the limit in which the chromosphere is a solid wall: $r \to \infty$. A shock is reflected back upward from this wall. What is the Mach number of this shock? What is the speed of the shock front relative to the contact discontinuity?

   c. The material behind this shock is at rest and has been shocked *twice*: once by the initial downward shock and once by the reflected shock. What is its density? What is its pressure?

   d. Assume now that r is large enough that the twice-shocked pressure is the same as in part b. There is, however, a *transmitted* shock propagating into the chromosphere. What is the Mach number of this shock in terms of r and v_f/c_{s0}?

   e. What is the flow velocity in the region between transmitted and reflected shocks? Which direction is this flow (upward or downward)?

[2] This idealized exercise may be considered a means of understanding the importance of conduction by ignoring it. Observations show the flare shock drives chromospheric material upward — this is called *chromospheric evaporation.*

3. It is possible to determine the properties of a shock using measurements of the magnetic field $\mathbf{B}$, the proton density n, and the velocity $\mathbf{v}$ relative to the spacecraft. This must be done without assuming knowledge of the shock's normal $\hat{\mathbf{n}}$ or speed v_s. Of course the problem is complicated considerably by the presence of numerous fluctuations on top of what would otherwise be a simple shock. Sophisticated methods have been developed to pull the shock properties out of the fluctuating data. Here we demonstrate, using ideal data, the principle with calculations simple enough to perform on a hand calculator. Two sets of ideal measurements have been made by a spacecraft moving radially outward from the Sun (the R direction) at $v_0 = 20$ km/sec. Velocities are expressed in the (R, T, N) coordinate system, relative to the spacecraft.

	17:25UT (first)	17:49UT (second)
n [cm^{-3}]	11.62	4.25
v_R [km/s]	349.5	295.9
v_T [km/s]	64.80	79.14
v_N [km/s]	37.98	59.75
B_R [nT]	0.027	1.000
B_T [nT]	-1.759	-0.908
B_N [nT]	-2.931	-1.474

a. Which of the measurements, first or second, is the pre-shock (upstream) region? Is the radial component of the shock normal, $\hat{\mathbf{n}} \cdot \hat{\mathbf{r}}$, positive or negative?

b. Using the jump condition $[\![B_n]\!] = 0$ show that the shock normal $\hat{\mathbf{n}}$ must be orthogonal to $[\![\mathbf{B}]\!]$.

c. Use the jump condition

$$\rho v_n [\![\mathbf{v}_\perp]\!] = \frac{B_n}{4\pi} [\![\mathbf{B}_\perp]\!] \ ,$$

to show that the velocity jump $[\![\mathbf{v}]\!]$ must lie in the plane spanned by $\hat{\mathbf{n}}$ and $[\![\mathbf{B}]\!]$. This condition is known as *co-planarity*. Does it matter which reference frame $\mathbf{v}$ is measured in to apply this principle?

d. Use the facts established in parts b. and c. to compute the shock normal $\hat{\mathbf{n}}$, from the data. This will naturally be expressed as components in (R, T, N) coordinates. How do you choose between 2 options?

e. Use $\hat{\mathbf{n}}$ and the magnetic field vectors to compute θ_1 and θ_2.

f. Use the two relations

$$\frac{n_2}{n_1} = \frac{M_{A1}^2}{M_{A2}^2} \quad , \quad [\![(M_A^2 - 1)\tan\theta]\!] = 0 \ ,$$

to find the Alfvén Mach numbers M_{A1} and M_{A2} for this shock.

g. What kind of shock is this? Fast, slow or intermediate?

h. Use the definition of the Alfvén Mach number

$$M_A = \frac{v_n}{B_n/\sqrt{4\pi\rho}} \ ,$$

to compute the velocity of the shock *relative to the spacecraft.*