Solution to Problem 1

(a) Since ρ is constant and uniform, the continuity equation reduces to

$$\nabla \bullet \mathbf{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0 .$$

which is satisfied when the expression for u_x and u_y are substituted into the equation.

(b) Since **B** does not vary in time, Faraday's equation reduces to

$$\nabla \times \mathbf{E} = \mathbf{0} .$$

Thus,

$$\frac{dE_z}{dx} = 0 \qquad \text{and} \qquad \frac{dE_z}{dy} = 0$$

These conditions are both satisfied if E_z = constant = $-E_0$.

(c) From Ampère's Law, the current density is

$$\mathbf{j} = -\frac{1}{\mu_0} \frac{\partial B_x}{\partial y} \hat{\mathbf{z}} ,$$

Substitution into Ohm's Law along with the expressions for **u** and **B**, yields

$$\frac{\partial B_x}{\partial y} + \frac{k\mu_0}{\eta_e} y B_x = \frac{E_0 \mu_0}{\eta_e} ,$$

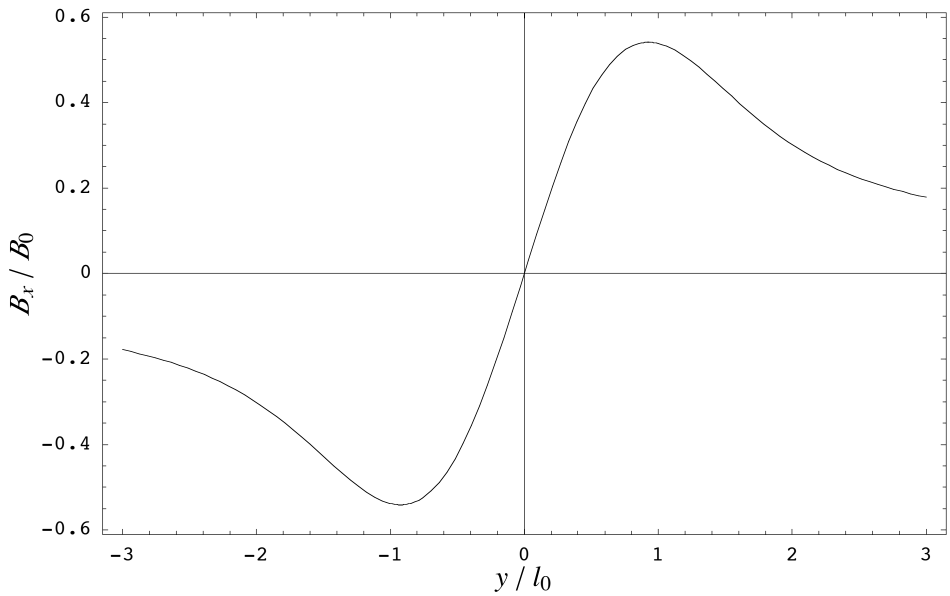
which is a first-order, linear , ordinary differential equation (ODE) with the solution

$$B_x = \frac{E_0 \mu_0 l_0}{\eta_e} e^{-(y/l_0)^2} \int_0^{y/l_0} e^{t^2} dt = \frac{E_0 \mu_0 l_0}{\eta_e} \operatorname{daw}(y/l_0) ,$$

where $l_0 = \sqrt{2\eta_e / k\mu_0}$ and daw(y/l_0) is the Dawson Integral function. Other notations used for this function are

$$\operatorname{daw}(x) = D_+(x) = \frac{1}{2}\sqrt{\pi} e^{-x^2} \operatorname{erfi}(x) = -\frac{1}{2} i \sqrt{\pi} e^{-x^2} \operatorname{erf}(ix) ,$$

(d)



The constant l_0 is the diffusive scale length, that is, the location where the outward diffusion of the magnetic field roughly equals the inward motion of the plasma. The constant B_0 is approximately 1.85 times the maximum value of B_x. This maximum value occurs at $y/l_0 = 0.924$ which is of order unity. Thus B_0 is about twice the maximum value of the magnetic field, and this maximum occurs close to the location where the outward diffusion of the field is balanced by the compression of the field due to the plasma inflow.

(e) The two components of the momentum equation are

$$u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} = -\frac{\partial p}{\partial y} - \frac{B_x}{\mu_0}\frac{\partial B_x}{\partial y} \ ,$$

and

$$u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} = -\frac{\partial p}{\partial x} \ .$$

Substitution of the expressions for u_x and u_y, leads to

$$-k^2 x = \frac{\partial p}{\partial x} \ , \quad \text{and} \quad -k^2 y = \frac{\partial}{\partial y}\left(p + \frac{B_x^2}{2\mu_0} \right)$$

which are both satisfied when

$$p = p_0 - \frac{1}{2} k^2 (x^2 + y^2) - \frac{1}{2} \frac{B_x^2}{2\mu_0}$$

where B_x is given by the solution to part (c). At large distance p tends to minus infinity primarily because the kinetic energy increases as distance squared. Thus, the solution is only valid for finite distances from the stagnation point at $x = 0$, $y = 0$.

Reconnection — solutions

4. (a) We find the field along the x axis by substituting $w = x$ into eq. (1). The result is purely real so $B_x = 0$ — the field is purely vertical. The vertical component is

$$B_y(x,0) \;=\; \frac{\mu_0 I_0/2\pi}{x-a} + \frac{\mu_0 I_0/2\pi}{x+a} \quad , \tag{5}$$

which vanishes when $x = 0$. The field is oriented downward ($B_y < 0$) through the surface from $x = 0$ to $x = a - r$ (the inside of the right wire), so the net flux is given by the integral

$$\psi_0 \;=\; -\int_0^{a-r} B_y(x,0)\,dx \;=\; \frac{\mu_0 I_0}{2\pi}\ln(a/2r) \quad . \tag{6}$$

(b) We evaluate the field on the x-axis by substituting $w = x$ into eq. (2)

$$B_y + iB_x \;=\; \frac{a\mu_0 I_0/\pi}{\sqrt{a^2-L^2}}\frac{\sqrt{x^2-L^2}}{(x^2-a^2)} \tag{7}$$

Outside the current sheet $x^2 > L^2$ the radical in the numerator is purely real and the field is therefore perfectly vertical: $B_x = 0$. On the other hand, within the current sheet $x^2 < L^2$ the radical in the numerator is purely imaginary and the field is therefore perfectly horizontal ($B_y = 0$).

To expand the $\sqrt{\cdot}$ about the branch point write $w = L + \epsilon\, e^{i\phi}$, where $\epsilon \ll L$ and ϕ is the polar angle. Expansion of (2) yields

$$B_y + iB_x \;\simeq\; -\frac{a\mu_0 I_0/\pi}{(a^2-L^2)^{3/2}}\sqrt{2L\epsilon}\, e^{i\phi/2} \quad . \tag{8}$$

This shows that to the right of the current sheet ($\phi = 0$) the field is directed downward — consistent with its sense just to the left of the right wire. Noting that $e^{\pm i\pi/2} = \pm i$, we see that $B_x < 0$ above the sheet ($\phi = +\pi$) and $B_x > 0$ below the sheet ($\phi = -\pi$) — consistent with the counter-clockwise sense the field has at great distances. The tangential component, B_x, is discontinuous across the sheet and there is no normal component there, $B_y = 0$; this is a *tangential discontinuity.*

(c) Using $|x| \le L \ll a$ in eq. (7), and the signs determined above, gives

$$B_x(x,\pm 0) \;=\; \mp\frac{\mu_0 I_0}{\pi a^2}\sqrt{L^2-x^2} \quad , \tag{9}$$

adjacent to the sheet. The peak field strength occurs at $x = 0$ where

$$B_{\rm pk} \;=\; \max|B_x(x,0)| \;=\; \frac{\mu_0 I_0 L}{\pi a^2} \quad . \tag{10}$$

Evaluating eq. (2) along the y axis ($w = iy$) gives

$$B_y + iB_x \;=\; \mp i\frac{a\mu_0 I_0/\pi}{\sqrt{a^2-L^2}}\frac{\sqrt{y^2+L^2}}{(y^2+a^2)} \;\simeq\; \mp i\frac{\mu_0 I_0}{\pi a^2}\sqrt{y^2+L^2} \quad , \tag{11}$$

where the final expression uses $y, L \ll a$. The field is purely horizontal ($B_y = 0$) and has the magnitude

$$|B_x| \;\simeq\; B_{\rm pk}\left(1 + \frac{y^2}{L^2}\right)^{1/2} \quad , \tag{12}$$

after using expression (10). This agrees with eq. (5.21) of Vol. I, after making the substitutions $B_{\rm pk} \to B_i$. Since the variable L matches in both expressions the full width of the sheet is $2L$, per our definition, rather than what is indicated on on figure 5.4.

(d) The total current is found by integrating expression (9) in a right-handed loop around the current sheet — under the sheet ($y = 0-$) from $x = -L$ to $+L$, then over the sheet ($y = +0$) from $x = +L$ to $-L$

$$I_{\rm cs} \;=\; \frac{1}{\mu_0}\oint \mathbf{B}\cdot d\mathbf{l} \;=\; \frac{2}{\mu_0}\int\limits_{-L}^{L} B_x(x,-0)\,dx \;\simeq\; I_0\frac{L^2}{a^2} \quad . \tag{13}$$

This current has the same sense at I_0. For very large complex coordinates, $|w| \gg a$, the complex field is

$$B_y + iB_x \;\sim\; \frac{a\mu_0 I_0/\pi}{\sqrt{a^2 - L^2}}\frac{1}{w} \;=\; \frac{\mu_0 I_0}{\pi}\frac{1}{w}\left(1 - \frac{L^2}{a^2}\right)^{-1/2} \;\simeq\; \frac{\mu_0 I_0}{2\pi}\left(2 + \frac{L^2}{a^2}\right)\frac{1}{w} \quad . \tag{14}$$

Comparing to a single current, $B_y + iB_x = \mu_0 I/2\pi$ we see an additional contribution $I_0(L^2/a^2)$ from the current sheet.

(e) For positions outside the current sheet, $x > L$, expression (7) is purely real, so the field is purely vertical. The private flux is the integral of this field

$$\psi \;=\; -\int\limits_{L}^{a-r} B_y(x,0)\,dx \;=\; \frac{a\mu_0 I_0/\pi}{\sqrt{a^2 - L^2}}\int\limits_{L}^{a-r}\frac{\sqrt{x^2 - L^2}}{x^2 - a^2}dx \quad . \tag{15}$$

The addition of a line current $I_{\rm cs}$ to eq. (1) gives

$$B_y + iB_x \;=\; \frac{\mu_0 I_0/2\pi}{w - a} + \frac{\mu_0 I_0/2\pi}{w + a} \;+\; \frac{\mu_0 I_{\rm cs}}{2\pi w} \;=\; \frac{\mu_0 I_0}{2\pi}\,\frac{(2 + L^2/a^2)w^2 - L^2}{w(w^2 - a^2)} \quad . \tag{16}$$

The second expression shows that field crosses the x axis downward within $L/\sqrt{2} < x < a$. This is the private flux encircling the right wire, and it amounts to

$$\begin{aligned}\psi \;&=\; -\int\limits_{L/\sqrt{2}}^{a-r} B_y(x,0)\,dx \;=\; \frac{\mu_0 I_0}{2\pi}\ln(a/2r) - \frac{\mu_0 I_{\rm cs}}{2\pi}\ln(\sqrt{2}a/L) \\ &=\; \frac{\mu_0 I_0}{2\pi}\ln(a/2r) - \frac{\mu_0 I_{\rm cs}}{4\pi}\ln(2I_0/I_{\rm cs}) \;=\; \psi(I_{\rm cs}) \quad .\end{aligned} \tag{17}$$

where the final expression uses eq. (13) to replace $a/L = \sqrt{I_0/I_{\rm cs}}$.

(f) Performing the energy integral

$$\Delta W \;=\; -\int\limits_{\psi_0}^{\psi(I_{\rm cs})} I_{\rm cs}\,d\psi \;=\; -\,I_{\rm cs}\psi(I_{\rm cs}) \;+\; \int\limits_0^{I_{\rm cs}} \psi(I_{\rm cs})\,dI_{\rm cs}$$

$$\;=\; -I_{\rm cs}[\psi(I_{\rm cs})-\psi_0] \;-\; \frac{\mu_0}{4\pi}\int\limits_0^{I_{\rm cs}} I_{\rm cs}\ln(2I_0/I_{\rm cs})\,dI_{\rm cs} \tag{18}$$

$$\;=\; \frac{\mu_0 I_{\rm cs}^2}{8\pi}\ln(2I_0/I_{\rm cs}) \;-\; \frac{\mu_0 I_{\rm cs}^2}{16\pi} \quad . \tag{19}$$

Expression (18) follows from using (17), and the final expression from performing the definite integral.

(g) The constraint of conserved private flux is

$$\psi(I_{\rm cs}) \;-\; \psi_0 \;=\; \frac{\mu_0 I_0}{2\pi}\ln(a/a_0) - \frac{\mu_0 I_{\rm cs}}{4\pi}\ln(2I_0/I_{\rm cs}) \;=\; 0 \quad . \tag{20}$$

This places a relation between the wire location and the sheet's current

$$\frac{I_{\rm cs}}{2I_0}\ln(2I_0/I_{\rm cs}) \;=\; \ln(a/a_0) \quad . \tag{21}$$

The left hand side is positive since $I_{\rm cs}/I_0 > 0$, thus we see than the wires must be separated ($a > a_0$) to produce a horizontal current sheet.

Both the left wire and the current sheet exert a force on the right wire by their vertical field contributions. We sum these contributions of eq. (16) at $x = a$ to produce the "external" field,

$$B_y^{\rm (ext)} \;=\; \frac{\mu_0 I_0}{4\pi a} \;+\; \frac{\mu_0 I_{\rm cs}}{2\pi a} \quad , \tag{22}$$

which is upward ($B_y > 0$). The two currents thus exert a leftward force (per length)

$$F_x \;=\; -\,I_0 B_y^{\rm (ext)} \;=\; -\,\frac{\mu_0 I_0^2}{4\pi a} \;-\; \frac{\mu_0 I_0 I_{\rm cs}}{2\pi a} \quad , \tag{23}$$

(attraction to the other parallel currents). Integrating this force over the displacement of the wire, and doubling it to account for the left wire, gives the work

$$\Delta W \;=\; -2\int\limits_{a_0}^{a} F_x(a)\,da \;=\; \frac{\mu_0 I_0^2}{2\pi}\ln(a/a_0) + \frac{\mu_0 I_0}{2\pi}\int\limits_{a_0}^{a}\frac{I_{\rm cs}\,da}{a}$$

$$\;=\; \frac{\mu_0 I_0^2}{2\pi}\ln(a/a_0) \;+\; \frac{\mu_0 I_0}{2\pi}I_{\rm cs}\ln(a/a_0) \;-\; \frac{\mu_0 I_0}{2\pi}\int\limits_0^{I_{\rm cs}}\ln(a/a_0)dI_{\rm cs} \tag{24}$$

$$\;=\; \frac{\mu_0 I_0^2}{2\pi}\ln(a/a_0) \;+\; \frac{\mu_0 I_{\rm cs}^2}{4\pi}\ln(2I_0/I_{\rm cs}) \;-\; \frac{\mu_0}{4\pi}\int\limits_0^{I_{\rm cs}} I_{\rm cs}\ln(2I_0/I_{\rm cs})\,dI_{\rm cs} \tag{25}$$

$$\;=\; \frac{\mu_0 I_0^2}{2\pi}\ln(a/a_0) \;+\; \frac{\mu_0 I_{\rm cs}^2}{8\pi}\ln(2I_0/I_{\rm cs}) \;-\; \frac{\mu_0 I_{\rm cs}^2}{16\pi} \quad . \tag{26}$$

Expression (25) follows from insertig constraint (21) into eq. (24). The first term in expression (26) is the work required to separate the wires without producing a current sheet — it would be necessary even if the wires were in a vacuum. The remainder is the additional work required due to the lack of reconnection to create new private flux. It matches the electromagnetic energy in eq. (19) — the work required to change the private flux.

5. The tangential component of the field at the top and bottom of the sheet is

$$B_x(x, \pm\delta) \;=\; \frac{\partial A}{\partial y} \;=\; \mp\,\frac{\mu_0 I_{\rm cs}}{\pi L}\sqrt{1 - x^2/L^2} \quad . \tag{27}$$

The normal component is

$$B_y(x, \pm\delta) \;=\; -\frac{\partial A}{\partial x} \;=\; \frac{1}{2}\frac{x\delta}{L^2 - x^2}|B_x| \quad . \tag{28}$$

It is evident that away from the tips, $L - |x| \gg \delta$, the field is predominantly tangential ($|B_x| \gg |B_y|$) and conforms to the Green-Syrovatskii form with peak strength

$$B_{\rm pk} \;=\; \frac{\mu_0 I_{\rm cs}}{\pi L} \quad , \tag{29}$$

achieves at the midpoint, $x = 0$.
The current density is

$$J_z(x, y) \;=\; -\frac{1}{\mu_0}\nabla^2 A \;\simeq\; -\frac{1}{\mu_0}\frac{\partial^2 A}{\partial y^2} \;=\; \frac{I_{\rm cs}}{\pi L\delta}\sqrt{1 - x^2/L^2} \;=\; \frac{|B_x(x, \delta)|}{\delta} \quad , \tag{30}$$

after noting that the $\partial^2 A/\partial x^2$ will be small provided $L - |x| \gg \delta$. The total current is found by integrating over the sheet

$$I \;=\; \int_{-L}^{L} dx \int_{-\delta}^{\delta} dy\, J_z(x, y) \;=\; \frac{2I_{\rm cs}}{\pi}\int_{-L}^{L}\sqrt{1 - x^2/L^2}\,\frac{dx}{L} \;=\; I_{\rm cs} \tag{31}$$

The Lorentz force density is

$$\begin{aligned} \mathbf{F} \;&=\; \mathbf{J}\times\mathbf{B} \;=\; \hat{\mathbf{y}}J_z(x)B_x(x, y) - \hat{\mathbf{x}}J_z(x)B_y(x, y) \\ &=\; -\frac{\mu_0 I_{\rm cs}^2}{\pi^2 L^2}\left[\left(1 - \frac{x^2}{L^2}\right)\frac{y}{\delta^2}\,\hat{\mathbf{y}} \;+\; \frac{y^2 x}{2\delta^2 L^2}\,\hat{\mathbf{x}}\right] \quad . \end{aligned} \tag{32}$$

The force is directed toward the origin. The largest forces are from the sides of the sheet near the mid-point, $x = 0$, $y = \pm\delta$ will begin to move the sides inward, making the sheet narrower and therefore making the current density *larger*. Since this is a Lorentz force it is directed so as to *reduce* the magnetic energy. Indeed, the minimum energy possible (without reconnection) is for a genuine magnetic discontinuity, $\delta \to 0$.

(b) The Alfvén speed at the position of peak field strength is

$$v_{\rm A,pk} \;=\; \frac{B_{\rm pk}}{\sqrt{\mu_0\rho}} \;=\; \frac{\mu_0 I_{\rm cs}}{\pi L\sqrt{\mu_0\rho}} \quad , \tag{33}$$

after using eq. (29). Using this in the Lunquist number gives

$$L_u \;=\; \frac{\mu_0 v_{\rm A,pk} L}{\eta_e} \;=\; \frac{\mu_0^{3/2} I_{\rm cs}}{\pi\eta_e\sqrt{\rho}} \;=\; \frac{I_{\rm cs}}{\pi\eta_e\sqrt{\rho}\,\mu_0^{-3/2}} \quad . \tag{34}$$

The denominator is a current

$$I_{\rm sp} = \frac{\pi\eta_e\sqrt{\rho}}{\mu_0^{3/2}} = \frac{\pi}{(4\pi\times 10^{-7})^{3/2}}\,\frac{\sqrt{m_i m_e}k_{\rm B}^{1/2}\ln\Lambda}{1.07\times 10^9\,e^2}\sqrt{\frac{n_i}{T_e^3}} = 10^{-2}\ln\Lambda\sqrt{\frac{n_i}{T_e^3}} \quad , \tag{35}$$

after using eqs. (5.12) and (5.13) from Vol. I for η_e. Using values from table 5.1 gives *very* small currents.

	magnetosphere	corona	
n_i	10^5	10^{15}	m^{-3}
T_e	10^7	10^6	K
$\ln\Lambda$	11	33	—
$I_{\rm sp}$	10^{-9}	10^{-2}	Amps

This means a magnetospheric current sheet carrying $I_{\rm cs} = 1$ Amp has $L_u \sim 10^9$ — a very large Lunquist number, meaning resistivity is a very small effect. A similar current sheet in the corona would have only a modest Lunquestion number ($L_u \sim 100$), but this is an exceptionally small current for such a big place. Typical coronal currents are $\gg 10^9$ Amps.

(c) The electric field at the center is

$$E_z(0) = \eta J_z(0) = \frac{\eta I_{\rm cs}}{\pi L\delta} \quad . \tag{36}$$

The electromagnetic work is

$$E_z I_{\rm cs} = \frac{\eta I_{\rm cs}^2}{\pi L\delta} \quad . \tag{37}$$

This can be seen to match the rate at which magnetic free energy is released form the current sheet equilibrium by time differentiating eq. (19). The power from direct Ohmic dissipation is

$$\begin{aligned} P_\eta &= \int \eta J^2\,dx\,dy = 2\eta\delta\int_{-\Delta}^{\Delta} J^2(x)\,dx = \frac{4\eta I_{\rm cs}^2}{(\pi L^2)^2\delta}(L^2\Delta - \tfrac{1}{3}\Delta^3) \\ &= \frac{4\eta I_{\rm cs}^2}{\pi^2 L\delta}\left(\frac{\Delta}{L} - \frac{\Delta^3}{3L^3}\right) = \frac{4}{\pi}\left(\frac{\Delta}{L} - \frac{\Delta^3}{3L^3}\right)E_z(0)I_{\rm cs} \quad . \end{aligned} \tag{38}$$

When the resistivity is uniform over the current sheet, $\Delta = L$ (Sweet-Parker reconnection) the magnetic energy released is converted mostly into heat directly by Ohmic dissipation. When the resistivity is concentrated in a small central region, $\Delta \ll L$, the resistivity thermalizes only a small fraction, $\Delta/L \ll 1$, directly. This corresponds to Petschek reconnection whereby slow magnetosonic shocks convert the remainder of the energy to kinetic energy and heat.

(d) The Poynting flux normal to the sides of the current sheet is

$$S_y(x,\pm\delta) = E_z B_x(x,\pm\delta) = \mp\,\eta\frac{\mu_0 I_{\rm cs}^2}{\pi^2 L^2\delta}\sqrt{1 - x^2/L^2} \quad . \tag{39}$$

The contribution along the "ends" at $x = \pm\Delta$ is generally smaller by $\sim \delta/L$ and their length also smaller by $\sim \delta/\Delta$, so we will neglect them. The net energy flux into the sheet is therefore

$$\begin{aligned} P &\simeq \int_{-\Delta}^{\Delta} [S_y(x,-\delta) - S_y(x,+\delta)]\,dx \;=\; \eta\frac{2\mu_0 I_{\rm cs}^2}{\pi^2 L\delta}\int_{-\Delta}^{\Delta}\sqrt{1-x^2/L^2}\,\frac{dx}{L} \\ &= \eta\frac{2\mu_0 I_{\rm cs}^2}{\pi^2 L\delta}\left[\sin^{-1}(\Delta/L) \;+\; (\Delta/L)\sqrt{1-\Delta^2/L^2}\right] \end{aligned} \tag{40}$$

In the limit $L \to \Delta$ we recover a power matching eq. (37), showing that all the electromagnetic work done on the sheet is transmitted through the Poynting flux. The restricted sheet limit, $\Delta \ll L$, gives

$$P \;\simeq\; \eta\frac{4\mu_0 I_{\rm cs}^2}{\pi^2 L\delta}\frac{\Delta}{L} \quad , \tag{41}$$

matching the same limit in eq. (38). This shows that all the energy entering the inner resistive region sheet is Ohmically dissipated.