SOLUTIONS
Homework Exercise
Solar Convection and the Solar Dynamo

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Equation numbers in the Homework set are identified with an H, e.g. equation (H3).

PROBLEM 1: THERMAL WIND BALANCE

(a) The key here is to first divide equation (H1) by ρ and then compute the curl of the Coriolis term with the help of (H22) and (H23). You should also notice that

$$\mathbf{\Omega_0} = \Omega_0 \hat{\boldsymbol{z}} = \Omega_0 \left(\cos\theta \hat{\boldsymbol{r}} - \sin\theta \hat{\boldsymbol{\theta}} \right) \quad . \tag{1}$$

It looks ugly at first glance, but if you work it out, you'll find that there is only one surviving nonzero term:

$$\left[\boldsymbol{\nabla} \times (2\mathbf{\Omega_0} \times \mathbf{v}) \right] \boldsymbol{\cdot} \hat{\boldsymbol{\phi}} = 2\mathbf{\Omega_0} \boldsymbol{\cdot} \boldsymbol{\nabla} v_\phi \quad . \tag{2}$$

Since Ω_0 and λ are independent of z and ϕ, and since the sphere is periodic in ϕ, an average over longitude $<>$ gives

$$\langle 2\mathbf{\Omega_0} \boldsymbol{\cdot} \boldsymbol{\nabla} v_\phi \rangle = 2\mathbf{\Omega_0} \boldsymbol{\cdot} \boldsymbol{\nabla} \langle v_\phi \rangle = 2\lambda \mathbf{\Omega_0} \boldsymbol{\cdot} \boldsymbol{\nabla} \Omega \quad . \tag{3}$$

Using (H20), (H18), and the chain rule for differentiation, it's straightforward to show that

$$\boldsymbol{\nabla} \times \left(\frac{\boldsymbol{\nabla} P}{\rho} \right) = - \frac{\boldsymbol{\nabla} P \times \boldsymbol{\nabla} \rho}{\rho^2} \quad . \tag{4}$$

Averaging equation (4) over longitude and using the result in equation (3) then leads you to the thermal wind equation, equation (H3).

(b) The Ω countours would be cylindrical, so your sketch should have straight lines parallel to the rotation axis, with the value of Ω increasing as you get farther from the rotation axis.

(c) The Coriolis force associated with the differential rotation is

$$-2\rho \mathbf{\Omega_0} \times \left(\langle v_\phi \rangle \hat{\boldsymbol{\phi}} \right) = 2\rho \Omega_0 \langle v_\phi \rangle \hat{\boldsymbol{\lambda}} \quad . \tag{5}$$

If $\Omega = \Omega(\lambda)$ then $\langle v_\phi \rangle$ is also a function of λ alone, so

$$2\rho\Omega_0 \langle v_\phi \rangle \hat{\boldsymbol{\lambda}} = \rho \boldsymbol{\nabla} \Psi_{DR} \tag{6}$$

where

$$\Psi_{DR} = 2\Omega_0 \int \langle v_\phi(\lambda)\rangle \, d\lambda \quad . \tag{7}$$

If we define $\Psi' = \Psi + \Psi_{DR}$, and if we neglect the terms on the left hand side (assuming a steady state and low Rossby number), then equation (H1) yields

$$\boldsymbol{\nabla} \langle P\rangle = \rho \boldsymbol{\nabla} \Psi' \quad . \tag{8}$$

Thus, constant $\langle P\rangle$ and Ψ' surfaces coincide because $\boldsymbol{\nabla} \langle P\rangle$ and $\boldsymbol{\nabla}\Psi'$ are parallel.

(d) Taking the gradient of equation (H4) gives

$$\frac{\boldsymbol{\nabla} S}{C_P} = \frac{\boldsymbol{\nabla} P}{\gamma P} - \frac{\boldsymbol{\nabla}\rho}{\rho} \tag{9}$$

So,

$$\frac{\boldsymbol{\nabla}\rho}{\rho} = \frac{\boldsymbol{\nabla} P}{\gamma P} - \frac{\boldsymbol{\nabla} S}{C_P} \quad . \tag{10}$$

and

$$\frac{\boldsymbol{\nabla} P \times \boldsymbol{\nabla}\rho}{\rho^2} = -\frac{\boldsymbol{\nabla} P \times \boldsymbol{\nabla} S}{\rho C_p} \quad . \tag{11}$$

The $\hat{\boldsymbol{\phi}}$ component of the right-hand side is

$$\frac{1}{r\rho C_P}\left(\frac{\partial P}{\partial r}\frac{\partial S}{\partial \theta} - \frac{\partial P}{\partial \theta}\frac{\partial S}{\partial r}\right) \tag{12}$$

Since the radial component of $\boldsymbol{\nabla} P$ is generally much larger than the latitudinal component, and since both components of $\boldsymbol{\nabla} S$ are comparable, then the first term in equation (12) will be much bigger than the second term. Using equation (H5) and plugging this into (H3) yields (H6).

Along with the usual approximations that justify the fluid equations (small mean-free path, thermodynamic equilibrium), our derivation of (H6) relied on these assumptions

1. We neglected the Lorentz force and viscous diffusion in equation (H1)
2. We assumed low Rossby number (the Coriolis force dominates the advection term)
3. We assumed a statistically steady state
4. We assumed an ideal gas equation of state with constant C_P, γ
5. We assumed that the stratification is nearly adiabatic and hydrostatic

The most essential approximations are 1 and 2, which gives us the force balance in equation (H3). Magnetic tension, meridional Reynolds stresses, advection of zonal vorticity by the meridional circulation, and viscous diffusion (in rough order of importance) can all disrupt thermal wind balance (there is also a potentially important component of the centrifugal force proportional to $\langle v_\phi\rangle^2$ that we have neglected as part of the low Rossby number approximation). A steady state (4) is not essential; $\partial \mathbf{v}/\partial t$ may be neglected based only on the low

Rossby number assumption. The ideal gas approximation (5) is not essential; we could have used an arbitrary equation of state $S(\rho, P)$ and we would still obtain (H6) but C_P^{-1} would be replaced by $\rho^{-1}(\partial\rho/\partial S)|_P$. Approximation 5 merely allowed us to simplify the right-hand side a bit and identify $\partial \langle S \rangle / \partial\theta$ as the most significant thermal gradient involved in thermal wind balance. Without this approximation, we would have the more general right-hand side as expressed in equation (H3).

(e) Helioseismic rotational inversions imply that $\partial\Omega/\partial z < 0$ in the northern hemisphere of the solar convection zone and $\partial\Omega/\partial z > 0$ in the southern hemisphere. According to (H6), $\partial \langle S \rangle / \partial\theta$ has the same sign, implying a poleward $\boldsymbol{\nabla} S$. If we attribute this entropy gradient to a temperature gradient, this implies *warm* (high entropy) poles. However, the effect is small and thus difficult to detect, yielding expected thermodynamic variations of about 10^{-5} relative to the hydrostatic background state. Current helioseismic structure inversions are not sensitive enough to confirm or deny such small thermal gradients.

PROBLEM 2: DYNAMO WAVES

(a) There's not much to say here beyond the hints. The "uncurled" version of equation (H7) is

$$\frac{\partial \mathbf{A}'}{\partial t} = \mathbf{v}\times\boldsymbol{B} + \alpha\boldsymbol{B} \tag{13}$$

where $\mathbf{A}'$ is the total vector potential, defined such that $\boldsymbol{\nabla}\times\mathbf{A}' = \boldsymbol{B}$. Note that A is the x component of $\mathbf{A}'$ and B_x comes from the y and z components:

$$B_x = \frac{\partial A'_z}{\partial y} - \frac{\partial A'_y}{\partial z} \quad . \tag{14}$$

We won't really need to use A'_y and A'_z in what follows; we just need to know that they are independent of x, because B_x is. Still, we'll find that it is sometimes useful to write $\boldsymbol{B}$ as $\boldsymbol{\nabla}\times\mathbf{A}'$ in order to use vector identities.

Since $\mathbf{v}$ is in the $\hat{\boldsymbol{x}}$ direction, the first term on the right-hand-side of (13) is perpendicular to the x direction. So, the x component of (13) gives (H10).

To derive (H11), first plug in (H8) for $\mathbf{v}$ to find that

$$\mathbf{v}\times\boldsymbol{B} = -\Gamma z B_z \hat{\boldsymbol{y}} + \Gamma z B_y \hat{\boldsymbol{z}} \quad . \tag{15}$$

Taking the x component of the curl then gives

$$\left[\boldsymbol{\nabla}\times(\mathbf{v}\times\boldsymbol{B})\right]\cdot\hat{\boldsymbol{x}} = \Gamma B_z + \Gamma z \left\{ \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} \right\} = -\Gamma \frac{\partial A}{\partial y} \quad . \tag{16}$$

The term in curly brackets is zero because $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$. Now note that $\boldsymbol{\nabla}\cdot(A\hat{\boldsymbol{x}}) = 0$ because A is independent of x. So, equation (H21) implies

$$\boldsymbol{\nabla}\times\left[\boldsymbol{\nabla}\times(A\hat{\boldsymbol{x}})\right] = -\nabla^2(A\hat{\boldsymbol{x}}) = -\nabla^2 A\ \hat{\boldsymbol{x}} \quad . \tag{17}$$

Plugging (16) and (17) into (H7) yields (H11).

(b) Neglecting the α term in equation (H11) gives

$$\frac{\partial B_x}{\partial t} = -\Gamma \frac{\partial A}{\partial y} \quad . \tag{18}$$

Then differentiating (H10) with respect to time yields

$$\frac{\partial^2 A}{\partial t^2} = \alpha \frac{\partial B_x}{\partial t} = -\alpha\Gamma \frac{\partial A}{\partial y} \quad , \tag{19}$$

where we have used (18).

(c) Plugging the expansion (H13) into (H12) and dividing by $\tilde{A}$ gives

$$\omega^2 = \imath\alpha\Gamma k \quad . \tag{20}$$

Taking the square root gives

$$\omega = \pm\sqrt{\frac{\alpha\Gamma k}{2}}\,(\imath + 1) = \pm\sqrt{\frac{s|\alpha\Gamma|k}{2}}\,(\imath + 1) \quad . \tag{21}$$

If $s = 1$ then $\sqrt{s} = 1$ and if $s = -1$ then $\sqrt{s} = \imath$. With a little thought you can convince yourself that (H14) satisfies both possibilities.

If you were to double α the wave would go faster in the same direction by a factor of $\sqrt{2}$. The exponential growth rate would also increase by a factor of $\sqrt{2}$. If you change the sign of α but not the amplitude, then the wave would go the other direction at the same speed. The same statements also apply to Γ if you were to keep α constant. In the Solar envelope, $\partial\Omega/\partial r$ is biggest near the base of the convection zone and tachocline. The predicted sign of α there (opposite to the sign of kinetic helicity) implies equatorward propagation.

(d) If the α term dominates in (H11) then instead of (H12) you'd get

$$\frac{\partial^2 A}{\partial t^2} = -\alpha^2 \nabla^2 A \quad . \tag{22}$$

Plugging in the expansion (H13) then gives

$$\omega^2 = -\alpha^2\left(k^2 + k_z^2\right) \tag{23}$$

where k_z is the vertical wavenumber, assuming $\tilde{A}(z) \propto \exp ik_z z$. Taking the square root gives

$$\omega = \pm\imath\alpha\sqrt{k^2 + k_z^2} \quad . \tag{24}$$

Equation (H13) then implies that there is no oscilltory behavoir for A; it either grows or decays exponentially, keeping the same spatial profile.

The implication is that shear, or in other words, differential rotation, helps promote dynamo waves, and thus cyclic activity, by introducing a phase shift between the poloidal and toroidal source terms. In short, an α-Ω dynamo would be more likely to exhibit a magnetic cycle than an α^2 dynamo. If this simple model is valid for the Sun, it suggests that differential rotation may have a key role to play in establishing the solar cycle.

(e) The full magnetic induction equation with diffusion is

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla}\times(\mathbf{v}\times\boldsymbol{B} + \alpha\boldsymbol{B} - \eta\boldsymbol{\nabla}\times\boldsymbol{B}) \tag{25}$$

Since $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$, equation (H21) implies

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{B}) = -\nabla^2\mathbf{B} \quad . \tag{26}$$

Also, since $\mathbf{A}'$ is independent of x, equation (H21) also yields

$$[\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A}')]\cdot\hat{\boldsymbol{x}} = -\nabla^2 A \quad . \tag{27}$$

Thus, in both equations (H10) and (H11) we can replace the $\partial/\partial t$ operator by

$$\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t} - \eta\nabla^2 \quad . \tag{28}$$

Plugging in the expansion of (H13) and applying this operator twice gives the substitution

$$\left[-\imath\omega^2\right] \rightarrow \left[-\imath\omega + \eta\left(k^2 + k_z^2\right)\right]^2 \tag{29}$$

...so

$$\omega^2 \rightarrow \left[\omega + \imath\eta\left(k^2 + k_z^2\right)\right]^2 \tag{30}$$

Plug this into (20) and you'll find that the dispersion relation (H14) is replaced by

$$\omega = \pm\sqrt{\frac{|\alpha\Gamma|k}{2}}\;(s + \imath) - \imath\eta\left(k^2 + k_z^2\right) \quad . \tag{31}$$

So, essentially, there is an extra term in the dispersion relation for ω that is purely imaginary and proportional to η.

A look at our expansion in equation (H13) shows that in order to have an exponentially growing solution, we need the imaginary part of ω to be positive. So, we'll take the positive root of (31) and require that

$$\sqrt{\frac{|\alpha\Gamma|k}{2}} - \eta\left(k^2 + k_z^2\right) > 0 \quad . \tag{32}$$

A little straightforward manipulation of this gives the condition $\mathcal{D} > 1$, where $\mathcal{D}$ is defined by (H17).