Martin Lee Solutions (2010)

In many of the problems encountered in the heliospheric transport of energetic particles, particles are scattered effectively in pitch-angle during timescales of interest. The scattering is due to the irregular electromagnetic fluctuations in the plasma that have a secular effect on the particle velocity. Under these circumstances the particle distribution functions can be assumed to be nearly isotropic, and the appropriate transport equation is the energetic particle transport equation first derived by Parker (*Planet. Space Sci.*, *13*, 9, 1965). Applications of this transport equation have had a huge impact on this area of research from the solar modulation of galactic cosmic rays, to the transport of solar energetic particles and the mechanism of diffusive shock acceleration. It is therefore essential for a student of energetic particle transport to gain familiarity with the equation, the physics behind it, and illustrative applications of the equation to many of the important energetic particle populations in the heliosphere. The problems that follow are rather diverse and only ordered by their difficulty with the easiest problems presented first.

1. Particle Conservation

Consider the Parker transport equation

$$\frac{\partial f}{\partial t} + (\mathbf{V} + \mathbf{V}_D) \cdot \nabla f - \nabla \cdot \mathbf{K} \cdot \nabla f - \frac{1}{3}(\nabla \cdot \mathbf{V})\, p \frac{\partial f}{\partial p} = 0$$

with no source of particles on the right hand side, where the drift velocity

$$\mathbf{V}_D = \frac{pvc}{3e} \nabla \times \frac{\mathbf{B}}{B^2}$$

Show explicitly that the total number of particles in phase-space is conserved as long as $f(|\mathbf{x}| \to \infty)$ and $f(p \to \infty)$ vanish. It is helpful to write the equation in conservation (continuity) form.

1. (Solution)

Since $\nabla \cdot \mathbf{V}_D = 0$, rewrite the equation as

$$\frac{\partial f}{\partial t} + \nabla \cdot \left[\left(\mathbf{V} + \mathbf{V}_D \right) f \right] - (\nabla \cdot \mathbf{V}) f - \frac{1}{3} (\nabla \cdot \mathbf{V}) p \frac{\partial f}{\partial p} - \nabla \cdot \mathbf{K} \cdot \nabla f = 0$$

Combine terms 3 and 4 to yield

$$\frac{\partial f}{\partial t} + \nabla \cdot \left[\left(\mathbf{V} + \mathbf{V}_D \right) f \right] - \frac{1}{p^2} \frac{\partial}{\partial p} \left[\frac{1}{3} (\nabla \cdot \mathbf{V}) p^3 f \right] - \nabla \cdot \mathbf{K} \cdot \nabla f = 0$$

Integrate the equation by operating with $\int d^3\mathbf{x} d^3\mathbf{p} = \int d^3\mathbf{x} 4\pi p^2 dp$, noting that $N = \int d^3\mathbf{x} d^3\mathbf{p} f$.

$$\frac{\partial N}{\partial t} + \int 4\pi p^2 dp \int d^3\mathbf{x} \nabla \cdot \left[\left(\mathbf{V} + \mathbf{V}_D \right) f - \mathbf{K} \cdot \nabla f \right] - \int d^3\mathbf{x} \int 4\pi p^2 dp \frac{1}{p^2} \frac{\partial}{\partial p} \left[\frac{1}{3} (\nabla \cdot \mathbf{V}) p^3 f \right] = 0$$

Note that the integrals only involve either $f\left(|x| \to \infty\right)$ or $f\left(p \to \infty\right)$, which vanish. This implies that $dN/dt = 0$, and that N is constant.

2. Interplanetary Propagation of Solar Energetic Particles (SEPs)

High-energy particles are accelerated close to the Sun in association with flares and coronal mass ejections (CMEs). They occur either as discrete impulsive events or gradual events. The former events are thought to be accelerated as a byproduct of magnetic reconnection at the flare site, while the latter events are thought to be accelerated at the shocks driven by fast CMEs near the Sun. In both cases these particles propagate into interplanetary space after their release at the Sun. The particles that arrive first at an observing spacecraft propagate nearly scatter-free through the ambient electromagnetic fields. However, those that arrive later have been scattered by electromagnetic fluctuations, have nearly isotropic velocity distributions, and may be described very approximately by the Parker transport equation.

The simplest possible model neglects particle drift, advection with the solar wind and adiabatic deceleration in the diverging wind. If *N* particles of a specific momentum magnitude p_0 are released impulsively at the Sun with spherical symmetry, they then satisfy

$$\frac{\partial f}{\partial t} = K(p_0)\nabla^2 f + \frac{N}{4\pi p^2}\delta(p - p_0)\delta(\mathbf{r})\delta(t)$$

where we have assumed that the diffusion tensor is isotropic and homogeneous. Find $f(r,t,p_0)$. For an observer at heliocentric radius *r*, at what time is the maximum particle intensity observed?

2. (Solution)

Take the Fourier transform of $f(x,y,z,t)$

$$-i\omega\tilde{f} = -K(k_x^2 + k_y^2 + k_z^2)\tilde{f} + N(4\pi p_0^2)^{-1}\delta(p - p_0)$$

$$\Rightarrow \tilde{f} = iN\left(4\pi p_0^2\right)^{-1}\delta\left(p - p_0\right)\left[\omega + iK\left(k_x^2 + k_y^2 + k_z^2\right)\right]^{-1}$$

$$f = \frac{1}{(2\pi)^4}\int d^3\mathrm{k}\, e^{i\mathrm{k}\cdot\mathrm{r}}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\,\frac{iN(4\pi p_0^2)^{-1}\,\delta(p - p_0)}{\omega + iK(k_x^2 + k_y^2 + k_z^2)}$$

Performing the ω-integration by closing the contour in the lower-half plane for $t > 0$, we obtain

$$f = \frac{1}{(2\pi)^4}\int d^3\mathrm{k}\, e^{i\mathrm{k}\cdot\mathrm{r}}(-2\pi i)iN(4\pi p_0^2)^{-1}\;\delta(p - p_0)e^{-Kk^2 t}$$

$$f = \frac{1}{(2\pi)^3}\frac{N}{4\pi p_0^2}\delta(p - p_0)\left[\int_{-\infty}^{\infty} dk_x e^{ik_x x} e^{-Kk_x^2 t}\right][y][z]$$

where the last pair of brackets are the same as the first but with *y* or *z* replacing *x*.

Completing the square in the exponent of the k_x-integration yields

$$f = \frac{1}{(2\pi)^3}\frac{N}{4\pi p_0^2}\delta\left(p - p_0\right)\left(\frac{\pi}{Kt}\right)^{1/2} e^{-x^2(4Kt)^{-1}}\left(\frac{\pi}{Kt}\right)^{1/2} e^{-y^2(4Kt)^{-1}}\left(\frac{\pi}{Kt}\right)^{1/2} e^{-z^2(4Kt)^{-1}}$$

$$= \frac{N}{4\pi p_0^2}\delta(p - p_0)\frac{1}{(4\pi Kt)^{3/2}}e^{-r^2(4Kt)^{-1}}$$

Maximum intensity $\Rightarrow \dfrac{\partial f}{\partial t} = 0 \Rightarrow t = \dfrac{r^2}{(6K)}$

3. The Solar Modulation of Galactic Cosmic Rays

Consider a simple model for the solar modulation of galactic cosmic rays, which nevertheless includes many of the important features of the process. Take the stationary spherically-symmetric Parker transport equation for constant solar wind speed *V* and $K_{rr} = Vr/2$ (independent of energy)

$$V\frac{\partial f}{\partial r} - \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{V}{2}r\frac{\partial f}{\partial r}\right) - \frac{1}{3}\frac{2V}{r}p\frac{\partial f}{\partial p} = 0$$

where drift transport is neglected. Find $f(r < r_0, p)$ subject to the boundary condition $f(r_0, p) = p_0\delta(p - p_0)$. The solution represents the modulation of a monoenergetic population of galactic cosmic rays. A more general energy spectrum of cosmic rays in interstellar space may be obtained by convolution. Hint: a more convenient choice of independent variables is $x = \ln(r/r_0)$ and $y = \ln(p/p_0)$. Describe the essential features of the solution. Find p_m, the momentum at which *f* has its maximum value, as a function of *r*.

3. (Solution)

$$r\frac{\partial f}{\partial r}-\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{2}r^3\frac{\partial f}{\partial r}\right)-\frac{2}{3}p\frac{\partial f}{\partial p}=0$$

$$r\frac{\partial f}{\partial r}-\frac{1}{2}r\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right)-r\frac{\partial f}{\partial r}-\frac{2}{3}p\frac{\partial f}{\partial p}=0$$

$$\frac{\partial^2 f}{\partial x^2}+\frac{4}{3}\frac{\partial f}{\partial y}=0$$

Fourier transform in y

$$\frac{\partial^2 \tilde{f}}{\partial x^2}-\frac{4}{3}i\omega\tilde{f}=0$$

$$\tilde{f}=C(\omega)\exp\left[\pm(4\mathrm{i}\omega/3)^{1/2}x\right]$$

Require convergence as $x\to-\infty$ (keep the sign which converges)

$$f=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega e^{-i\omega y}C(\omega)\,\mathbf{exp}\left[\pm(4i\omega/3)^{1/2}x\right]$$

Since $f(x=0,y)=\delta(y)\Rightarrow C(\omega)=1$. There is a branch point at $\omega=0$.

Since $f(y>0)=0$ (particles only lose energy), we choose the integration path to pass below $\omega=0$ and the branch cut along the positive imaginary axis.

For $y>0$, we complete the contour with an infinite semi-circle in the lower half-plane. We obtain

$$f(y>0)=0$$

For $y<0$ we evaluate the integral along a contour descending down the left side of the branch cut and ascending up the right side.

On the left side $\omega=\xi e^{-i3\pi/2}$, $d\omega=d\xi e^{-3i\pi/2}$

On the right side $\omega=\xi e^{i\pi/2}$, $d\omega=d\xi e^{i\pi/2}$

$$f=\frac{1}{2\pi}\int_{\infty}^{0}d\xi e^{-i3\pi/2}e^{-iy\xi i}\,\mathbf{exp}\left\{\pm\left[4\,\mathbf{exp}(i\pi/2)\xi\,\mathbf{exp}(-i3\pi/2)/3\right]^{1/2}x\right\}$$

$$+\frac{1}{2\pi}\int_0^\infty d\xi e^{i\pi/2} e^{-iy\xi i}\mathbf{exp}\left\{\pm\left[4\,\mathbf{exp}(i\pi/2)\xi\,\mathbf{exp}(i\pi/2)/3\right]^{1/2} x\right\}$$

With $\omega = \xi e^{i\theta} \Rightarrow \sqrt{ie^{i\theta}} = \mathbf{exp}\left[i(\theta/2+\pi/4)\right]$

$$\frac{-\pi}{2} < \frac{\theta}{2} + \frac{\pi}{4} < \frac{\pi}{2}$$

Thus the real part is convergent with the (+) sign for $x < 0$

$$f = \frac{i}{2\pi}\int_0^\infty d\xi e^{y\xi}\left\{\mathbf{exp}\left[ix(4\xi/3)^{1/2}\right] - \mathbf{exp}\left[-ix(4\xi/3)^{1/2}\right]\right\}$$

$$= -\frac{1}{\pi}\int_0^\infty d\xi e^{y\xi}\,\mathbf{sin}\left[x(4\xi/3)^{1/2}\right]$$

$$= \frac{1}{\pi}\int_0^\infty d\xi \exp(-|y|\xi)\sin\left[|x|(4\xi/3)^{1/2}\right] = \frac{1}{\pi}\int_0^\infty 2\eta d\eta \exp(-|y|\eta^2)\sin\left[\eta|x|(4/3)^{1/2}\right]$$

$$= \frac{1}{\pi}\int_{-\infty}^\infty \eta d\eta\,\mathbf{exp}(-|y|\eta^2)\,\mathbf{sin}\left[\eta|x|(4/3)^{1/2}\right]$$

$$= -\frac{\sqrt{3}}{2\pi}\frac{\partial}{\partial|x|}\int_{-\infty}^\infty d\eta\,\mathbf{exp}(-|y|\eta^2)\,\mathbf{cos}\left[\eta|x|2(3)^{-1/2}\right]$$

$$= -\frac{\sqrt{3}}{2\pi}\frac{\partial}{\partial|x|}\int_{-\infty}^\infty d\eta \exp(-|y|\eta^2)\exp\left[\mathrm{i}\eta|x|2(3)^{-1/2}\right]$$

$$= -\frac{\sqrt{3}}{2\pi}\frac{\partial}{\partial|x|}\int_{-\infty}^\infty d\eta \exp\left\{-|y|\left[\eta - i|x||y|^{-1}(3)^{-1/2}\right]^2\right\}\exp\left[-|x|^2(3|y|)^{-1}\right]$$

$$= -\frac{\sqrt{3}}{2\pi}\frac{\partial}{\partial|x|}\exp\left[-|x|^2(3|y|)^{-1}\right]\underbrace{\int_{-\infty}^\infty d\eta' e^{-|y|\eta'^2}}_{\frac{1}{|y|^{1/2}}\sqrt{\pi}}$$

$$= -\frac{\sqrt{3}}{2}\frac{1}{\pi}\frac{\sqrt{\pi}}{|y|^{1/2}}\mathbf{exp}\left[-|x|^2(3|y|)^{-1}\right]\left(-\frac{2|x|}{3|y|}\right)$$

$$= \frac{1}{\sqrt{3\pi}} \frac{|x|}{|y|^{3/2}} \mathbf{exp}\left[-|x|^2 \left(3|y|\right)^{-1}\right]$$

To calculate the value of $p = p_m$, where $f(p)$ is a maximum:

$$\frac{\partial f}{\partial p} = 0 = \frac{-1}{p} \frac{\partial f}{\partial |y|}$$

$$\Rightarrow -\frac{3}{2} \frac{1}{|y|^{5/2}} e^{-\frac{|x|^2}{3|y|}} + \frac{1}{|y|^{3/2}} e^{-\frac{|x|^2}{3|y|}} \frac{|x|^2}{3|y|^2} = 0$$

$$\frac{3}{2} = \frac{|x|^2}{3|y|}$$

$$|y| = \frac{2}{9}|x|^2 = -\mathbf{ln}\frac{p_m}{p_0}$$

$$\frac{p_m}{p_0} = \exp\left\{-(2/9)\left[\ln\left(r/r_0\right)\right]^2\right\}$$

4. A Simple Model for the Production and Evolution of Interstellar Pickup Ions in the Solar Wind.

Interstellar gas enters the heliosphere under the influence of solar gravity, radiation pressure, and ionization losses. The resulting neutral atom density is $n(r,\theta)$, where *r* is heliocentric radial distance and θ is the angle of the heliocentric position vector relative to the bulk inflow velocity of the atoms. We may assume that the ionization rate per atom is $\beta_0(r_0/r)^2$. When an atom is ionized it has a speed approximately equal to the solar wind speed *V* in the frame of the solar wind. We assume that these ions are immediately picked up by the solar wind via gyration and pitch-angle scattering to form an isotropic shell of speed *V* in the solar wind frame.

(a) Assuming that the pitch-angle scattering rate is so large that the spatial diffusion tensor is negligible, write down the Parker equation for the evolution of the pickup ion omnidirectional distribution function $f(r,\theta,v)$ with an appropriate source term. We assume that the configuration is stationary and that the solar wind has constant speed and spherical symmetry.
(b) Solve the Parker equation for $f(r,\theta,v)$.
(c) Approximate $f(r,\theta,v)$ for large *r*.
(d) Draw a schematic plot of $f(r,\theta,v)$ versus *v*.

4. (Solution)

(a) $$\frac{\partial f}{\partial t} + \mathbf{V}\cdot\nabla f - \frac{1}{3}\nabla\cdot\mathbf{V}\,v\frac{\partial f}{\partial v} = \beta_0\left(\frac{r_0}{r}\right)^2 n(r,\theta)\frac{\delta(v-V)}{4\pi v^2}$$

When integrated over $d^3\mathrm{v}$, the RHS gives the rate of pickup ion generation by ionization:

$$\frac{\partial n_{pi}}{\partial t} = \beta_0\left(\frac{r_0}{r}\right)^2 n(r,\theta)$$

Under the specified assumptions

$$\frac{\partial f}{\partial r} - \frac{2}{3r}v\frac{\partial f}{\partial v} = \beta_0\left(\frac{r_0}{r}\right)^2 n(r,\theta)\frac{\delta(v-V)}{4\pi V^3}$$

(b) Solution by characteristic curves:

$$dr = \frac{dv}{-\frac{2}{3}r^{-1}v} = \frac{df}{\beta_0\left(\frac{r_0}{r}\right)^2 n(r,\theta)\frac{\delta(v-V)}{4\pi V^3}}$$

The family of characteristic curves is given by:

$$\frac{dr}{r} = -\frac{3}{2}\frac{dv}{v} \Rightarrow rv^{3/2} = C$$

(information travels along these curves)

f = constant along a characteristic curve for $v < V$ and for $v > V$:

for $v > V$: $f = 0$ since particles lose energy by adiabatic cooling

for $v < V$: $f = f(C)$, constant along each characteristic curve

$$\frac{\partial f}{\partial v} = -\frac{3}{2}\frac{r}{v}\beta_0\left(\frac{r_0}{r}\right)^2 n(r,\theta)\frac{\delta(v-V)}{4\pi V^3}$$

where $r = Cv^{-3/2}$

$$\frac{\partial f}{\partial v} = -\frac{3}{2}\frac{1}{v}\beta_0 r_0^2 \frac{v^{3/2}}{C} n(Cv^{-3/2},\theta)\frac{\delta(v-V)}{4\pi V^3}$$

Integrating across $v = V$

$$f(V+\varepsilon) - f(V-\varepsilon) = -\frac{3}{2}\beta_0 r_0^2 \frac{V^{1/2}}{C} n(CV^{-3/2},\theta)\frac{1}{4\pi V^3}$$

$$\Rightarrow f(C) = \frac{3}{2}\frac{\beta_0 r_0^2 V^{1/2}}{C} n(CV^{-3/2},\theta)\frac{1}{4\pi V^3}$$

Thus

$$f(r, v < V) = \frac{3}{2}\frac{\beta_0 r_0^2 V^{1/2}}{r v^{3/2}} n\left(r\frac{v^{3/2}}{V^{3/2}},\theta\right)\frac{1}{4\pi V^3}$$

For $v \ll V$, $n(rv^{3/2}V^{-3/2},\theta)$ is exponentially small due to ionization.

(c) For large r and $v \sim V$, n approaches the interstellar number density so that $f(r, v < V) \propto r^{-1}v^{-3/2}$.

(d) A schematic plot of $f(v)$ versus v should be drawn.

5. Diffusive Acceleration at a Planar Stationary Shock

Consider particle acceleration and transport at a planar stationary shock at $x = 0$, for which the Parker transport equation in the shock frame is

$$V_x \frac{\partial f}{\partial x} - \frac{\partial}{\partial x}\left(K \frac{\partial f}{\partial x}\right) - \frac{1}{3}\frac{dV_x}{dx} p \frac{\partial f}{\partial p} = 0 \qquad (1)$$

The upstream fluid flow is $V_x(x < 0) = V_u > 0$ and the downstream fluid flow is $V_x(x > 0) = V_d > 0$, where both V_u and V_d are constants. The diffusion coefficients are $K(x < 0) = K_u$ and $K(x > 0) = K_d$, where K_u and K_d are functions only of p. The boundary conditions are that $f(x \to \infty)$ is finite and $f(x \to -\infty) = f_\infty(p)$, where $f_\infty(p)$ represents the ambient population of energetic particles. The objective of this problem is to calculate $f(x,p)$.

(a) Solve equation (1) separately upstream ($x < 0$) and downstream ($x > 0$) of the shock. Each solution should involve two undetermined functions of p.
(b) Impose the boundary conditions at $x \to -\infty$ and $x \to \infty$.
(c) Impose the condition $f(x = -\varepsilon) = f(x = \varepsilon)$ at the shock as $\varepsilon \to 0$. Why is this condition appropriate?
(d) The final undetermined function of p is determined by integrating equation (1) from $x = -\varepsilon$ to $x = \varepsilon$ and allowing ε to approach zero. This “jump condition” yields a first-order differential equation for the remaining unknown function. What is the physical meaning of this jump condition? Solve the differential equation to determine the function.
(e) Write out $f(x < 0,p)$ and $f(x > 0,p)$ explicitly.
(f) Evaluate $f(x,p)$ for the specific case $f_\infty(p) = f_0 \delta(p - p_0)$.
(g) In this case write the power-law index in terms of the shock compression ratio $\rho_d / \rho_u = V_u / V_d \equiv X$, where ρ is the fluid mass density.

5. (Solution)

(a) For $x < 0$ and $x > 0$

$$V_x \frac{\partial f}{\partial x} - K \frac{\partial^2 f}{\partial x^2} = 0$$

A trial solution is $f \propto \mathbf{exp}(\alpha x) \Rightarrow \alpha V_x - K\alpha^2 = 0 \Rightarrow \alpha = 0$ or $\alpha = V_x / K$.

$x < 0$ **:** $f = A(p)\,\mathbf{exp}(V_u x / K_u) + B(p)$

$x > 0$ **:** $f = C(p)\exp\left(V_d x / K_d\right) + D(p)$

(b) $x \to -\infty : f = f_\infty(p) \Rightarrow B(p) = f_\infty(p)$

$x \to \infty : f$ finite $\Rightarrow C(p) = 0$

(c) $f(x,p)$ must be continuous at the shock. A discontinuity would imply an infinite flux for a diffusive process, which in turn would smooth the discontinuity and lead to continuity.

$f(x = -\varepsilon) = f(x = \varepsilon) \Rightarrow A(p) + f_\infty(p) = D(p)$ where $\varepsilon \to 0$.

(d) Note that $V_x\, \partial f / \partial x$ is finite near $x = 0$; integration of this term vanishes as $\varepsilon \to 0$. Integrating the equation yields:

$$-K \frac{\partial f}{\partial x}(x = \varepsilon) + K \frac{\partial f}{\partial x}(x = -\varepsilon) - \frac{1}{3}\left[V_x(\varepsilon) - V_x(-\varepsilon)\right] p \frac{\partial f}{\partial p}(x = 0) = 0$$

This condition represents the continuity of the differential flux of particles at the shock including both diffusive and convective parts.

Since $\partial f / \partial x = 0$ for $x > 0$, we obtain

$$K_u\left(\frac{V_u}{K_u}\right) A - \frac{1}{3}(V_d - V_u)\, p \frac{\partial D}{\partial p} = 0$$

$$\Rightarrow \frac{1}{3}(V_u - V_d)\, p \frac{\partial D}{\partial p} + V_u (D - f_\infty) = 0$$

$$\Rightarrow \frac{\partial D}{\partial p} + \frac{3V_u}{V_u - V_d} \frac{D}{p} = \frac{3V_u}{V_u - V_d} \frac{f_\infty(p)}{p}$$

Let $\dfrac{3V_u}{V_u - V_d} \equiv \beta$

The integrating factor is $\exp(\beta \int dp\, p^{-1}) = p^\beta$

$$\frac{d}{dp}(p^{\beta}D) = \beta p^{\beta - 1} f_{\infty}(p)$$

$$p^{\beta}D = \beta \int_0^p dp'(p')^{\beta - 1} f_{\infty}(p') + D_0$$

Since the particles accelerated must depend on the magnitude of $f_{\infty}(p)$, and as $p \to 0$ there can be no accelerated particles, $D_0 = 0$ (also $D_0 \neq 0$ would result in a divergence as $p \to 0$)

$$\Rightarrow D(p) = \beta \int_0^p \frac{dp'}{p'} \left(\frac{p'}{p}\right)^{\beta} f_{\infty}(p')$$

(e) $x < 0: f = \left[D(p) - f_{\infty}(p)\right] \exp\left(V_u x / K_u\right) + f_{\infty}(p)$

$x > 0: f = D(p)$

(f) $f_{\infty}(p) = f_0 \delta(p - p_0)$

$\Rightarrow D(p < p_0) = 0$

$$D(p > p_0) = \beta \frac{f_0}{p_0} \left(\frac{p_0}{p}\right)^{\beta}$$

(g) $\beta = \dfrac{3V_u}{V_u - V_d} = \dfrac{3}{1 - \frac{V_d}{V_u}} = \dfrac{3}{1 - \frac{1}{X}} = \dfrac{3X}{X - 1}$

6. A Simple Example of a Shock Modified by Energetic Particle Pressure.

Consider a fluid with mass density ρ, velocity **V**, and negligible pressure. It transports nonrelativistic energetic particles, which are coupled to it by a constant diffusion coefficient *K*. The relevant equations are the hydrodynamic equations for the fluid and the Parker equation for the energetic particles (ignoring the magnetic field):

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{V}) = 0, \tag{1}$$

$$\rho\frac{\partial \mathbf{V}}{\partial t} + \rho(\mathbf{V}\cdot\nabla)\mathbf{V} = -\nabla P, \tag{2}$$

$$\frac{\partial f}{\partial t} + \mathbf{V}\cdot\nabla f - K\nabla^2 f - \frac{1}{3}\nabla\cdot\mathbf{V} v\frac{\partial f}{\partial v} = 0, \tag{3}$$

where *P* is the energetic particle pressure $\left(P = (4\pi/3)\int dv m v^4 f\right)$

(a) Take the pressure moment of equation (3) to derive an equation for $P(\mathbf{x},t)$. You should get a factor 5/3; set $\gamma = 5/3$.

(b) Now consider a stationary planar system with variations in the *x*-direction only. Rewrite equations (1) and (2), and the equation for $\partial P/\partial t$ derived in (a) specifically for this system.

(c) Find three integrals of the system and identify them as mass flux, momentum flux and energy flux conservation. Identifying the integral associated with the *P* equation is somewhat tricky. Rewrite the factor PdV/dx appearing in one term as $d/dx(PV) - VdP/dx$. Then in the terms involving the derivative dP/dx use the simplified version of equation (2) to replace dP/dx by the term in equation (2) involving *V* and *dV*/*dx*. The resulting equation may be integrated easily.

(d) Determine the three constants by setting $V = V_0 > 0$, $\rho = \rho_0$ and $P = 0$ as $x \to -\infty$.

(e) Derive the following equation for $V(x)$ alone by eliminating *P* in the energy flux integral:

$$\frac{2K}{\gamma+1}\frac{dV}{dx} = (V - V_0)\left(V - \frac{\gamma-1}{\gamma+1}V_0\right)$$

(f) Solve this equation for $V(x)$ and interpret the constant of integration. Derive expressions for $\rho(x)$ and $P(x)$. Plot all three functions schematically. Think about the derived structure. What does the solution represent?

6. (Solution)

(a) Pressure moment of equation (3)

$$\frac{\partial P}{\partial t} + \mathbf{V}\cdot \nabla P - K\nabla^2 P - \frac{1}{3}\nabla\cdot \mathbf{V}\frac{1}{3}4\pi \underbrace{\int dv\, mv^5 \frac{\partial f}{\partial v}}_{-5\int dv\, mv^5 f} = 0$$

$$\Rightarrow \frac{\partial P}{\partial t} + \mathbf{V}\cdot\nabla P - K\nabla^2 P + \frac{5}{3}(\nabla\cdot \mathbf{V})P = 0$$

Set $\gamma \equiv 5/3$

(b) For a stationary planar system

$$\frac{d}{dx}(\rho V) = 0$$

$$\rho V\frac{dV}{dx} = -\frac{dP}{dx}$$

$$V\frac{dP}{dx} - K\frac{d^2P}{dx^2} + \gamma\frac{dV}{dx}P = 0$$

(c) $\rho V = A$, mass flux

$\rho V^2 + P = B$, momentum flux

For the last equation write

$$V\frac{dP}{dx} - K\frac{d^2P}{dx^2} + \gamma\frac{d}{dx}(PV) - \gamma V\frac{dP}{dx} = 0$$

$$\Rightarrow -(\gamma-1)V\frac{dP}{dx} - K\frac{d^2P}{dx^2} + \gamma\frac{d}{dx}(PV) = 0$$

$$\Rightarrow (\gamma-1)\rho V^2\frac{dV}{dx} - K\frac{d^2P}{dx^2} + \gamma\frac{d}{dx}(PV) = 0$$

$$\Rightarrow \rho V\frac{V^2}{2} - \frac{K}{\gamma-1}\frac{dP}{dx} + \frac{\gamma}{\gamma-1}PV = C, \text{ energy flux}$$

(note that $P/(\gamma-1) = \varepsilon$, energy density of energetic particles)

(d) $A = \rho_0 V_0$

$B = \rho_0 V_0^2$

$C = \frac{1}{2}\rho_0 V_0^3$

(e) $\rho V \frac{V^2}{2} + \frac{K}{\gamma - 1}\rho V \frac{dV}{dx} + \frac{\gamma}{\gamma - 1} V\left(B - \rho V^2\right) = C$

$$\Rightarrow \frac{2K}{\gamma + 1}\frac{dV}{dx} = (V - V_0)\left(V - \frac{\gamma - 1}{\gamma + 1}V_0\right)$$

(f) Let $z = \frac{(\gamma + 1)x}{2K}$

$$dz = \frac{dV}{(V - V_0)\left(V - \frac{\gamma - 1}{\gamma + 1}V_0\right)}$$

$$= \frac{\gamma + 1}{2V_0}\left[\frac{dV}{V - V_0} - \frac{dV}{V - \frac{\gamma - 1}{\gamma + 1}V_0}\right]$$

$$z = \frac{\gamma + 1}{2V_0}\left[\ln\left(V_0 - V\right) - \ln\left(V - \frac{\gamma - 1}{\gamma + 1}V_0\right)\right] + C'$$

note $V < V_0$ and $\frac{dV}{dx} < 0$

C' describes the position of the structure in x; we take $C' = 0$

$$\frac{V_0}{K}x = \ln\frac{V_0 - V}{V - \frac{\gamma - 1}{\gamma + 1}V_0}$$

Solve for $V(x)$

$$V = V_0 \frac{1 + \frac{\gamma - 1}{\gamma + 1}\exp\left(V_0 x/K\right)}{1 + \exp\left(V_0 x/K\right)}$$

$$\rho = \rho_0 \frac{1+\exp(V_0 x/K)}{1+\frac{\gamma-1}{\gamma+1}\exp(V_0 x/K)}$$

$$P = \rho_0 V_0^2 - \rho V^2 = \rho_0 V_0 (V_0 - V)$$

$$= \rho_0 V_0^2 \frac{2}{\gamma+1} \frac{\exp(V_0 x/K)}{\left[1+\exp(V_0 x/K)\right]}$$

Approximate sketches of V, ρ and P should all be drawn.

The structure represents a strong shock modified by the energetic particles. Since $P(x \to -\infty) = 0$, the shock has infinite Mach number and a compression ratio of 4. The energetic particle acceleration provides all the shock dissipation. There is no fluid subshock.

7. Stochastic Acceleration of Particles in a Homogeneous Plasma.

Stochastic acceleration of particles is a classical acceleration mechanism. The original version of the mechanism, second-order Fermi acceleration, was developed by Fermi to account for the acceleration of galactic cosmic rays by "collisions" with interstellar "clouds." Although the original application of the mechanism is no longer viable, subsequent versions describe the acceleration of particles by a spectrum of Alfvén waves, by a spectrum of magnetosonic waves, by stochastic compressions and expansions in a plasma, and by multiple shock waves. The basic mechanism may be understood by considering the elastic scattering of particles off a homogeneous isotropic ensemble of massive spheres with random velocities **V**, radius *R*, and density *N*. The appropriate transport equation is

$$\frac{\partial f}{\partial t} = \frac{1}{p^2}\frac{\partial}{\partial p}\left[p^2 D(p)\frac{\partial f}{\partial p}\right]$$

where *f* is the omnidirectional distribution function, *p* is momentum magnitude, $D(p) = (1/3)\langle V^2\rangle\lambda^{-1}p^2/v$, *v* is particle speed, and $\lambda[=(\pi R^2 N)^{-1}]$ is the scattering mean free path. Calculate $f(t,p)$ if $f(0,p) = n_0(4\pi p_0^2)^{-1}\,\delta(p-p_0)$ and the particles are nonrelativistic. It is helpful to choose variables $P = p/p_0$ and τ, an appropriate dimensionless time. Find limiting forms for $f(P,\tau)$ for (a) $\tau \ll 1$ and *P* arbitrary and (b) $\tau \gg 1$ and *P* finite.

7. (Solution)

Since $p \propto v$, $D \propto p$, we have in dimensionless variables τ and P

$$\frac{\partial f}{\partial \tau} = \frac{1}{P^2}\frac{\partial}{\partial P}\left[P^3 \frac{\partial f}{\partial P}\right]$$

where $D = Pp_0^2 D_0$ and $\tau = D_0 t$. Laplace transforming in τ yields

$$P\frac{d^2\tilde{f}}{\partial P^2} + 3\frac{\partial \tilde{f}}{\partial P} - s\tilde{f} = -\frac{n_0}{4\pi}\delta(P-1)$$

the solution of the homogeneous equation is

$$\tilde{f}(P) = P^{-1} Z\left(2s^{1/2}P^{1/2}\right)$$

where $Z(z)$ is the modified Bessel function $I_2(z)$ or $K_2(z)$. $\tilde{f}(P)$ must be continuous at $P = 1$ and satisfy

$$P\frac{\partial \tilde{f}}{\partial P}\bigg|_{1-\varepsilon}^{1+\varepsilon} = -\frac{n_0}{4\pi}$$

To satisfy boundary conditions as $P \to 0,\ \infty$ we choose

$$\tilde{f}(P > 1) = AP^{-1} K_2\left(2s^{1/2}P^{1/2}\right)$$

$$\tilde{f}(P < 1) = BP^{-1} I_2\left(2s^{1/2}P^{1/2}\right)$$

The jump conditions yield

$$AK_2\left(2s^{1/2}\right) - BI_2\left(2s^{1/2}\right) = 0$$

$$AK'_2\left(2s^{1/2}\right)s^{1/2} - BI'_2\left(2s^{1/2}\right)s^{1/2} = -\frac{n_0}{4\pi}$$

where I'_2 is the derivative of I_2 with respect to the argument. The determinant of the coefficients is

$Det = W[I_2, K_2] = -s^{-1/2}/2$, where W is the Wronskian.

$$\Rightarrow A = n_0(2\pi)^{-1} I_2\left(2s^{1/2}\right)$$

$$B = n_0(2\pi)^{-1} K_2\left(2s^{1/2}\right)$$

$$\Rightarrow \tilde{f}(P>1) = n_0(2\pi P)^{-1} I_2\left(2s^{1/2}\right)K_2\left(2s^{1/2}P^{1/2}\right)$$

$$\tilde{f}(p<1) = n_0(2\pi P)^{-1} K_2\left(2s^{1/2}\right) I_2\left(2s^{1/2}P^{1/2}\right)$$

To perform the inverse transform we must calculate

$$I \equiv \int_L ds e^{s\tau} I_2\left(\alpha s^{1/2}\right)K_2\left(\beta s^{1/2}\right)$$

There is a branch point at $s = 0$. Let the branch cut be the negative real axis of *s*.

$$I = -\int_0^\infty d\xi e^{-\xi\tau} I_2\left(i\alpha\xi^{1/2}\right)K_2\left(i\beta\xi^{1/2}\right)$$

$$+\int_0^\infty d\xi e^{-\xi\tau} I_2\left(-i\alpha\xi^{1/2}\right)K_2\left(-i\beta\xi^{1/2}\right)$$

Now $I_2\left(\pm i\alpha\xi^{1/2}\right) = -J_2\left(\alpha\xi^{1/2}\right)$

$$K_2(i\beta\xi^{1/2}) - K_2(-i\beta\xi^{1/2}) = -i\pi I_2(i\beta\xi^{1/2})$$

$$\Rightarrow I = i\pi\int_0^\infty d\xi e^{-\xi\tau} J_2\left(\alpha\xi^{1/2}\right)J_2\left(\beta\xi^{1/2}\right)$$

Using integral tables

$$I = i\pi\tau^{-1} I_2\left(\frac{\alpha\beta}{2\tau}\right)\exp\left[-\frac{\left(\alpha^2+\beta^2\right)}{4\tau}\right]$$

$$\Rightarrow f = \frac{n_0}{2\pi P}\frac{1}{2\tau} I_2\left(\frac{2P^{1/2}}{\tau}\right)\exp\left(-\frac{\mathrm{P}+1}{\tau}\right)$$

For small τ and finite *P*

$$f \propto \tau^{-1/2}\exp\left[-\left(\mathrm{P}^{1/2}-1\right)^2/\tau\right]$$

For large τ and finite *P*

$$f \propto \tau^{-3}$$

8. Particle Scattering by a Magnetic Irregularity.

Consider the motion of a proton in a magnetic field given by

$$\mathbf{B} = B_0\left(\frac{dF}{dz}\mathbf{i} + \frac{dG}{dz}\mathbf{j} + \mathbf{k}\right)$$

where $F = F(z)$ and $G = G(z)$.

(a) Give the equations for $x(z)$ and $y(z)$ describing the magnetic field lines.

(b) Write down the three components of the equation of motion, $md^2\mathbf{r}/dt^2 = (e/c)(d\mathbf{r}/dt)\times\mathbf{B}$, involving d^2x/dt^2, d^2y/dt^2 and d^2z/dt^2.

(c) Show explicitly that the proton speed v is a constant.

(d) Integrate and manipulate the equations for d^2x/dt^2 and d^2y/dt^2 to show that if $F(z\to\pm\infty) = F_\pm$ and $G(z\to\pm\infty) = G_\pm$, where $F_\pm$ and $G_\pm$ are all constants, a proton that traverses the configuration from $z = -\infty$ to $z = +\infty$ encircles the same field line at $z = +\infty$ as it encircled at $z = -\infty$.

This means that in this configuration the particle precisely follows the field line.

(e) Now take

$$F(z) = \varepsilon\sin(2\pi z/L)\exp(-z^2/l^2)$$

$$G(z) = \varepsilon\cos(2\pi z/L)\exp(-z^2/l^2)$$

where $\varepsilon << 1$. Sketch a field line as carefully as you can.

(f) To zeroth order in ε, the proton trajectory satisfies $z = v_{z0}t$ and $\mathbf{v} = v_{\perp 0}\sin(\Omega t+\phi)\mathbf{i} + v_{\perp 0}\cos(\Omega t+\phi)\mathbf{j} + v_{z0}\mathbf{k}$, where $\Omega = e_pB_0/(m_pc)$. Integrate the equation for d^2z/dt^2 to calculate to order ε the change in v_z, Δv_z, as the proton moves from $z\to-\infty$ to $z\to+\infty$. You may wish to integrate by parts.

(g) Interpret your answer. Do you see evidence for the cyclotron resonance condition? What determines the sign of Δv_z?

8. (Solution)

$$\mathbf{B} = B_0\left(\frac{dF}{dz}\mathbf{i} + \frac{dG}{dz}\mathbf{j} + \mathbf{k}\right)$$

(a)
$$\frac{dx}{dF/dz} = \frac{dy}{dG/dz} = \frac{dz}{1}$$

$$\frac{dx}{dz} = \frac{dF}{dz} \Rightarrow x = F(z) + x_0$$

$$\frac{dy}{dz} = \frac{dG}{dz} \Rightarrow y = G(z) + y_0$$

(b)
$$m\,d^2\mathbf{r}/dt^2 = \frac{e}{c}\left(d\mathbf{r}/dt\right)\times\mathbf{B} = \frac{e}{c}B_0\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ dx/dt & dy/dt & dz/dt\\ dF/dz & dG/dz & 1\end{vmatrix}$$

$$= \frac{e}{c}B_0\left[\mathbf{i}\left(\frac{dy}{dt} - \frac{dz}{dt}\frac{dG}{dz}\right) - \mathbf{j}\left(\frac{dx}{dt} - \frac{dz}{dt}\frac{dF}{dz}\right) + \mathbf{k}\left(\frac{dx}{dt}\frac{dG}{dz} - \frac{dy}{dt}\frac{dF}{dz}\right)\right]$$

$$\frac{d^2x}{dt^2} = \Omega\left(\frac{dy}{dt} - \frac{dz}{dt}\frac{dG}{dz}\right)$$

$$\frac{d^2y}{dt^2} = -\Omega\left(\frac{dx}{dt} - \frac{dz}{dt}\frac{dF}{dz}\right)$$

$$\frac{d^2z}{dt^2} = \Omega\left(\frac{dx}{dt}\frac{dG}{dz} - \frac{dy}{dt}\frac{dF}{dz}\right)$$

(c)
$$\frac{d^2x}{dt^2}\frac{dx}{dt} + \frac{d^2y}{dt^2}\frac{dy}{dt} + \frac{d^2z}{dt^2}\frac{dz}{dt} = 0 \Rightarrow \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = v^2 = \text{const}$$

(d)
$$\frac{dx}{dt} = \Omega\left(y - G - C_y\right)$$

$$\frac{dy}{dt} = -\Omega(x - F - C_x)$$

When G and *F* are constant (as for $z \to \pm\infty$)

$$\frac{d^2x}{dt^2} = \Omega\frac{dy}{dt} = -\Omega^2(x - F - C_x)$$

$$\frac{d^2y}{dt^2} = -\Omega\frac{dx}{dt} = -\Omega^2(y - G - C_y)$$

This is a simple harmonic oscillator centered on the field line at $x = F + C_x$ and $y = G + C_y$.

Thus the particle encircles the same field line wherever the field line is straight, in particular as $z \to -\infty$ and $z \to +\infty$.

(e) Draw a sketch of the field line determined by $F(z)$ and $G(z)$.

(f) $$\frac{d^2z}{dt^2} = \frac{dv_z}{dt} = \Omega\left(\frac{dx}{dt}\frac{dG}{dz} - \frac{dy}{dt}\frac{dF}{dz}\right)$$

Since $F,G \sim \varepsilon$, the zeroth order orbit of the proton may be substituted on the right hand side

$$\frac{dv_z}{dt} = \Omega\left(\frac{dx/dt}{dz/dt}\frac{dG}{dt} - \frac{dy/dt}{dz/dt}\frac{dF}{dt}\right)$$

$$= \Omega\frac{v_{\perp 0}}{v_{zo}}\left[\sin(\Omega t + \phi)\frac{dG}{dt} - \cos(\Omega t + \phi)\frac{dF}{dt}\right]$$

$$\Delta v_z = \Omega\frac{v_{\perp 0}}{v_{zo}}\int_{-\infty}^{\infty} dt\left[\sin(\Omega t + \phi)\frac{dG}{dt} - \cos(\Omega t + \phi)\frac{dF}{dt}\right]$$

Integrate by parts noting that *G,F* vanish as $t \to \pm\infty$.

$$\Delta v_z = \Omega\frac{v_{\perp 0}}{v_{z0}}\int_{-\infty}^{\infty} dt\left[-\Omega\cos(\Omega t + \phi)G - \Omega\sin(\Omega t + \phi)F\right]$$

$$\Delta v_z = -\Omega^2\frac{v_{\perp 0}}{v_{z0}}\varepsilon\int_{-\infty}^{\infty} dt\left[\cos(\Omega t + \phi)\cos(2\pi v_{z0}t/L) + \sin(\Omega t + \phi)\sin(2\pi v_{z0}t/L)\right]\exp\left(-v_{zo}^2\ t^2/l^2\right)$$

$$= -\Omega^2\frac{v_{\perp 0}}{v_{z0}}\varepsilon\int_{-\infty}^{\infty} dt\cos\left[\left(\Omega - \frac{2\pi v_{z0}}{L}\right)t + \phi\right]\exp\left(-v_{zo}^2\ t^2/l^2\right)$$

$$\Delta v_z = -\Omega^2\frac{v_{\perp 0}}{v_{z0}}\varepsilon\,\mathrm{Re}\int_{-\infty}^{\infty} dt\exp\left\{i\left[\left(\Omega - 2\pi v_{z0}/L\right)t + \phi\right]\right\}\exp\left(-v_{z0}^2\ t^2/l^2\right)$$

$$= -\Omega^2 \frac{v_{\perp 0}}{v_{z0}} \varepsilon \mathrm{Re} \exp(\mathrm{i}\phi) \int_{-\infty}^{\infty} dt \exp\left\{ -\frac{v_{z0}^2}{l^2} \left[t - \frac{i}{2}\left(\Omega - \frac{2\pi v_{z0}}{L}\right)\frac{l^2}{v_{z0}^2} \right]^2 \right\} \exp\left[-\frac{l^2}{4 v_{z0}^2}\left(\Omega - \frac{2\pi v_{z0}}{L}\right)^2 \right]$$

$$= -\Omega^2 \frac{v_{\perp 0}}{v_{z0}} \varepsilon \mathrm{Re} \exp(i\phi) \frac{l}{v_{z0}} \int_{-\infty}^{\infty} d\tau \exp\left\{ - \left[\tau - \frac{\mathrm{i}}{2} \frac{v_{z0}}{l}\left(\Omega - \frac{2\pi v_{z0}}{L}\right)\frac{l^2}{v_{z0}^2} \right]^2 \right\} \exp\left[- \frac{l^2 \Omega^2}{4 v_{z0}^2}\left(1 - \frac{2\pi v_{z0}}{\Omega L}\right)^2 \right]$$

$$= -\Omega^2 \frac{v_{\perp 0}}{v_{z0}^2} \varepsilon l \cos\phi \sqrt{\pi} \exp\left[- \frac{l^2 \Omega^2}{4 v_{z0}^2}\left(1 - \frac{2\pi v_{z0}}{\Omega L}\right)^2 \right]$$

(g) The cyclotron resonance factor appears in parentheses: Unless $kv_{z0} - \Omega \cong 0$, the exponential factor makes Δv_z small. The "sharpness" of the resonance condition is dictated by l^2.

Clearly Δv_z is $\propto \varepsilon$ and $\propto l$.

The sign of Δv_z is determined by ϕ, the phase of the proton gyration relative to the phase of the magnetic field rotation.